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On the Local Times and Boundary Properties of Reflected Diffusions with Jumps in the Positive Orthant

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Abstract. In this paper we study boundary properties of reflected diffusions with positive and negative jumps, constrained to lie in the positive orthant of $\mathbf{R}^n$. We consider a model with oblique reflections and characterize the regulator processes in terms of semi-martingale local times at the boundary or reflection faces of $\mathbf{R}^n_+$. In particular, we show that under mild boundary conditions on the diffusion coefficients, and under a completely-S structure for the reflection matrix with an additional invertibility requirement, the regulator processes do not charge the set of times spent by the process at the intersection of two or more boundary faces. Other supporting results are also provided, as for example the fact that the law in $\mathbf{R}^n_+$ of the process at time t does not charge boundary faces for Lebesgue-a.e. t. The case of hyper-rectangular state spaces contained in the positive orthant is also considered.

Keywords: reflected diffusions, jumps, oblique reflection directions, regulator processes, local times, semi-martingales

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1. Introduction

Reflected diffusions with jumps arise in a wide variety of applications such as finance, queueing and risk theory, and models of manufacturing plants. For example, Kella and Whitt [8], Chen and Whitt [3] have shown that, in the heavy traffic limit, the process of the number of customers in an open queueing network subject to service interruptions can be approximated in a weak-convergence sense by reflected Brownian motions with jumps in the positive orthant. More recently, such processes have also been shown to arise in the context of heavy-tailed distributions, as for example random walk limits involving heavy-tailed step distributions, [17].

Reflected diffusion models with jumps are natural generalizations of the class of so-called piecewise deterministic Markov processes, [4]. The generalization being that the diffusive component adds to the randomness of the evolution of the process between jumps, and the reflections guarantee for the process to stay within a given region, as for example in queueing networks where the processes are non-negative. These models are also of interest in the risk and insurance context, where the jumps could be the claims while the diffusion arises due to volatility of the interest rates, for example. They also play an important role in the context of barrier options in mathematical finance.

A special case of reflected diffusions, namely semi-martingale reflecting Brownian motion, or SRBM for short, has been studied quite extensively due to its importance in models of queueing networks in heavy traffic, [5, 18, 19]. In [12] the authors established a boundary property in that the regulator processes do not charge the set of times spent by the SRBM at the intersection of two or more boundary faces. This property was then used for example in [14] to develop numerical methods for computing the stationary distribution (when it exists) of queueing networks in heavy traffic, or in [5] in the context of Brownian models of open queueing networks and the existence/uniqueness of their stationary distributions. For the corresponding existence/uniqueness of SRBMs in an orthant, see [15].

The authors considered one-dimensional reflected diffusions with jumps [10], showing for example that the law in $\mathbf{R}_+$ of the process at time t does not assign probability mass to the origin for Lebesgue-a.e. t, and characterizing the regulator process in terms of the corresponding semi-martingale local time at level zero. These boundary properties were then used to derive forward equations, and to study the stationary distributions of this class of one-dimensional reflected processes.

This paper is devoted to study boundary properties of reflected diffusions with possible time and space dependent drift and diffusion coefficients, as well as in the presence of possible signed jumps, constrained to lie in the positive orthant of $\mathbf{R}^n$. In particular, we will show that the boundary property in [12] continues to hold for this class of reflected diffusion models, but requiring, as a trade-off in our proof, an additional invertibility condition on the reflection matrix over just the completely-S structure considered for SRBMs in [12]. (In fact the authors also established there the necessity of this structure for such processes to exist.) In addition, we will generalize to the multi-dimensional setting the previously mentioned boundary properties in [10].

The organization of the paper is as follows. In Section 2 we introduce the model to be considered and obtain some preliminary results. In Section 3 we give the main results of the paper. In Section 4 we show how our results continue to hold in the case of hyper-rectangular state spaces contained in $\mathbf{R}_+^n$. Finally, in Section 5 we offer some further comments on the scope of the results presented in the paper.

2. Model formulation and preliminary results

Let $n \geq 2$ be an integer, $\mathbf{R}$ the set of real numbers, $\mathbf{R}^n = \times_{i=1}^n \mathbf{R}$, $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$, $\mathbf{R}_+^n = \times_{i=1}^n \mathbf{R}_+$ and $\mathbf{R}^{n\times n}$ the collection of all $n \times n$ real matrices. Also, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a stochastic basis satisfying the usual hypotheses, i.e., $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous. We consider the following reflected diffusion constrained to lie in $\mathbf{R}_+^n$ with positive and negative jumps:

$$X_t = X_0 + \int_0^t b(s, X_{s-})\, ds + \int_0^t \sigma(s, X_{s-})\, dW_s + \sum_{0<s\leq t} \Delta X_t + RZ_t \qquad (2.1)$$

where[1]:

- $X = (X_t)_{t\geq 0} = (X_t^1, \ldots, X_t^n)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted, $\mathbf{R}_+^n$-valued *càdlàg* semi-martingale, $X_{t-} = \lim_{s\uparrow t} X_s$ with $X_{0-} = 0$ by convention, and $\Delta X_t = X_t - X_{t-}$. We assume hereafter that $\sum_{0<s\leq t} |\Delta X_s| < \infty$ a.s. for each $t > 0$.
- $W = (W_t)_{t\geq 0} = (W_t^1, \ldots, W_t^n)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-standard Brownian motion on $\mathbf{R}^n$.
- $Z = (Z_t)_{t\geq 0} = (Z_t^1, \ldots, Z_t^n)_{t\geq 0}$ is a continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted, $\mathbf{R}_+^n$-valued process, with each Z^i non-decreasing, null at zero, and such that $\int_{\mathrm{R}_+} X_s^i\, dZ_s^i = 0$.
- $b = (b^i)_{i\in\{1,\ldots,n\}} : \mathbf{R}_+^{n+1} \to \mathbf{R}^n$ and $\sigma = (\sigma^{ij})_{i,j\in\{1,\ldots,n\}} : \mathbf{R}_+^{n+1} \to \mathbf{R}^{n\times n}$ are Borel measurable functions. We set $a = (a^{ij})_{i,j\in\{1,\ldots,n\}} \triangleq \sigma\sigma^T$, where σ^T corresponds to the transpose of matrix σ.

[1]Throughout the paper (in)equalities involving vectors or vector-valued processes are to be understood componentwise, "0" represents the appropriate null element clear from the context, "∞" represents $(\infty, \ldots, \infty)$ in a vectorial context and, even though vectors or vector-valued processes are written as row vectors, they are treated as column vectors in all equations they appear.

- $R = (R^{ij})_{i,j \in \{1,\dots,n\}} \in \mathbf{R}^{n \times n}$ is a completely-S matrix (described below), satisfying the following additional requirement: every principal submatrix of R is non-singular, i.e., for each $K \subsetneq \{1,\dots,n\}$ we have $R^{(K)}$ invertible, where $R^{(K)}$ denotes the principal submatrix obtained from R by deleting its kth row and column for all $k \in K$ (none if $K = \emptyset$).

The terminology completely-S is used in [12] to designate a real square matrix D having the property that, for each principal submatrix $\widetilde{D}$ of D, there exists a vector y with non-negative components (of the corresponding proper dimension), such that $\widetilde{D}y > 0$ (in particular D must have strictly positive diagonal elements). The authors established there the necessity of this structure on the reflection matrix for SRBMs in the positive orthant to exist. This is not unexpected since it accounts for the fact that, upon hitting the boundary of $\mathbf{R}^n_+$, the SRBM cannot leave the orthant.

The additional invertibility requirement on principal submatrices of R that we impose will allow us to relate the regulator processes Z^i's to semi-martingale local times, as well as to establish the boundary property in that the Z^i's do not charge the set of times spent by X at the intersection of two or more boundary faces of $\mathbf{R}^n_+$. Along with identifying the completely-S structure as the minimal necessary on the reflection matrix of SRBMs in the positive orthant, in [12] the authors also established the above boundary property for such class of processes under that minimal structure. Though our assumptions here on R are more restrictive because of the additional invertibility requirement on its principal submatrices, we believe the setting in (2.1) still encompasses a large class of models appearing in applications, in that the drift and diffusion coefficients are allowed to depend on time and space, as well as X is allowed to have signed jumps.

Note our requirements on R are satisfied for example by the class of P-matrices, i.e., matrices for which every principal minor is strictly positive, see [2]. (Examples of this last class are real triangular matrices with strictly positive diagonal elements or, more generally, positive definite matrices.) However, our requirements on R generate a class which strictly includes the class of P-matrices. Indeed, consider for example $n = 2$ and

$$R = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Then, R is completely-S and every principal submatrix obtained from it is invertible, but R is not a P-matrix.

In this paper we assume the existence of a tuple (X, Z), as already described, satisfying (2.1), i.e., with semi-martingale $X - X_0$ having canonical decomposition $X - X_0 = A + M + J$, $A_0 = M_0 = J_0 = 0$, where the continuous finite variation process A and the continuous local martingale term M are given by $A_\cdot = \int_0^\cdot b(s, X_{s-})\, ds + RZ_\cdot$ and $M_\cdot = \int_0^\cdot \sigma(s, X_{s-})\, dW_s$, respectively, and

$J_{\cdot} = \sum_{0<s\leq \cdot} \Delta X_s$ with $\sum_{0<s\leq t} |\Delta X_s| < \infty$ a.s. for each $t > 0$. This paper is not concerned with the existence issues. However, note for example (2.1), as an equation in (X, Z) for a given particular structure of the jumps, possesses unique strong solutions for each initial state $X_0 = x_0 \in \mathbf{R}_+^n$ when b and σ satisfy the usual local Lipschitz and linear growth conditions, and the following two conditions are satisfied.

1) The jumps are of the form

$$\sum_{0<s\leq t} \Delta X_s = \int_0^t \int_E \xi(s, X_{s-}, r)\, q(ds, dr),$$

where E is some Polish space (for example $\mathbf{R}$), q is an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure on $\mathbf{R}_+ \times E$ with intensity measure (or compensator) $\hat{q}(dt, dr) = \lambda G(dr)dt$, where the jump rate λ is finite and G is a probability measure on $(E, \mathcal{B}(E))$, $\xi = (\xi^i)_{i\in\{1,\dots,n\}} : \mathbf{R}_+^{n+1} \times E \to \mathbf{R}^n$ is Borel measurable and such that $\xi(t, x, r) \geq -x$ and each $\xi^i(t, x, r) \neq 0$ for each $(t, x, r) \in \mathbf{R}_+^{n+1} \times E$, and G and ξ satisfy the following. For each positive integer k, there exists $\rho_k : E \to \mathbf{R}_+$, such that $\int_E \rho_k^2(r)G(dr) < \infty$ and, for all $r \in E$, all $t \leq k$, and all $x, y \in \mathbf{R}_+^n$ with $\|x\|, \|y\| \leq k$,

$$\|\xi(t, x, r) - \xi(t, y, r)\| \leq \rho_k(r)\|x - y\|$$

and

$$\|\xi(t, x, r)\|^2 \leq \rho_k^2(r)(1 + \|x\|^2),$$

where $\|\cdot\|$ denotes the usual Euclidian norm in $\mathbf{R}^n$.

2) $R = I - Q$, where I is the identity matrix in $\mathbf{R}^{n\times n}$ and $Q \in \mathbf{R}^{n\times n}$ is such that $Q \geq 0$, elementwise, and its spectral radius is strictly less than one. (Note then R is a P-matrix.)

For more detailed treatments on the existence and uniqueness of solutions to reflecting stochastic differential equations with jumps, see for example [9,13]. (An excellent treatment for the case without reflections but in the presence of jumps can also be found in [7].)

Note that by writing $X^i = V^i + R^{ii}Z^i$ from equation (2.1), the regulator process Z^i of X^i at level zero can be characterized as follows [6, 16, 17]:

$$Z^i_{\cdot} = \frac{1}{R^{ii}} \sup_{s\in[0,\cdot]} \max\{-V^i_s, 0\}. \tag{2.2}$$

Of course, the above expression does not yield an explicit representation for Z since each V^i contains the components Z^j, $j \neq i$. (Also note since X is constrained to lie in $\mathbf{R}_+^n$, we have $X_0 \geq 0$ and $\Delta X_t \geq -X_{t-}$ for all $t > 0$, and therefore the right-hand side in (2.2) is continuous, in agreement with the definition of Z.)

The following notation will be used throughout the paper. We write, for any real-valued function $f(t, x^1, \ldots, x^n)$ with $t \in \mathbf{R}_+$ and $x = (x^1, \ldots, x^n) \in \mathbf{R}^n_+$, set $K \subseteq \{1, \ldots, n\}$ and vector $u \in \mathbf{R}^n_+$, $f(t, x^{u_K})$ to indicate that all x^i-arguments in f with indexes $i \in K$ are set to the values of the respective components in u. If $r \in \mathbf{R}_+$ (i.e., just a scalar), then by $f(t, x^{r_K})$ we indicate that all x^i-arguments in f with indexes $i \in K$ are set to the value r. In the same way, for u and v in $\mathbf{R}^n_+$, and sets $K_1, K_2 \subseteq \{1, \ldots, n\}$, $K_1 \cap K_2 = \emptyset$, we write $f(t, x^{u_{K_1}}, x^{v_{K_2}})$ to indicate that all x^i-arguments in f with indexes $i \in K_1$ are set to the values of the respective components in u, and similarly for $i \in K_2$ and v, with the same meaning as before in the case that u or v are just scalars. Of course, any of the previous index sets is allowed to be empty, in which case no corresponding setting of arguments takes place. For $i \in \{1, \ldots, n\}$ and $r \in \mathbf{R}_+$, we simply write $f(t, x^{r_i})$ for $f(t, x^{r_{\{i\}}})$. We use the previous notation in the following context. When we write for example $f(t, x^{u_K}) > 0$, we indicate that f at time t is strictly positive for all values of its remaining x-arguments, i.e., for all values of its x^i-arguments with $i \in K^c$, where K^c denotes the complement of K with respect to the index set $\{1, \ldots, n\}$. All of the above applies the same for $f(x^1, \ldots, x^n)$, i.e., for f not depending on t. Also, for $i \in \{1, \ldots, n\}$ we write $dx^{\neq i}$ to denote $dx^1 \cdots dx^n$ when the ith differential dx^i is omitted, $x^{\neq i}$ to denote $x \in \mathbf{R}^n_+$ when its ith coordinate x^i is omitted as well, and $X^{r_i}_t$ to denote X_t when its ith component X^i_t is replaced by $r \in \mathbf{R}_+$. Finally, $\mathbf{1}\{\cdot\}$ denotes the indicator function of the corresponding event in parentheses, $|K|$ the number of elements in set K, $K \setminus \widetilde{K}$ the usual set-theoretic difference, and m Lebesgue measure in $\mathbf{R}_+$.

As usual, whenever we write a.e. (almost everywhere, or almost every) without specifying the measure, we mean a.e. with respect to Lebesgue measure in the corresponding real space (clear from the context), and whenever we write a.s. (almost surely), we mean a.s. with respect to P.

Finally, we write P^X_t for the law of X_t in $\mathbf{R}^n_+$, $t \in \mathbf{R}_+$, $[X^i, X^i]^c$ for the path by path continuous part of the quadratic variation process $[X^i, X^i]$, and $(L^i(t, r))_{t,r \geq 0}$ for the jointly continuous in t and right continuous in r version of the local time ($r \in \mathbf{R}_+$ indicating the level) for semi-martingale X^i. Note since $\sum_{0<s\leq t} |\Delta X_s| < \infty$ a.s. for each $t > 0$, by [11, Theorem 56 and Corollary 3, p. 176 and 178, resp.] this version of the local time exists and, moreover, for each $(t, r) \in \mathbf{R}^2_+$ we have

$$L^i(t, r) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}\{r \leq X^i_s \leq r + \varepsilon\}\, d[X^i, X^i]^c_s \ \text{-a.s.}$$

Obviously it is enough for us to consider in this paper $r \geq 0$ only, since X is constrained to lie in the positive orthant of $\mathbf{R}^n$.

Before establishing some preliminary lemmas towards the main results of the paper to be given in the next section, we make the following remark.

Remark 2.1. Note since X is *càdlàg*, it can have at most a countable number of jumps in any compact interval of times contained in $\mathbf{R}_+$. Therefore, we can always replace X_{s-} by X_s in integrals of the form $\int_0^\cdot f(X_{s-})\mu(ds)$, and viceversa, when the measure μ is diffuse (i.e., when μ has no atoms). Note that is the case for Lebesgue measure and for the measures dZ_s^i and $L^i(ds,r)$ that Z_s^i and $L^i(s,r)$ induce in $\mathbf{R}_+$, respectively. This fact will be used from now on in the paper without further comment.

Lemma 2.1. *Let $t \in \mathbf{R}_+$ and $K \subseteq \{1,\dots,n\}$. Assume that there exists $i \in K$ such that $a^{ii}(s,x^{0_K}) > 0$ for a.e. $s \in [0,t]$. Then we have a.s.:*

$$m\{s \in [0,t] : X_s^k = 0, \forall k \in K\} = 0.$$

Also, for a.e. $s \in [0,t]$ we have:

$$P_s^X\{x \in \mathbf{R}_+^n : x^k = 0, \forall k \in K\} = 0$$

i.e., for a.e. $s \in [0,t]$, P_s^X does not charge the set $\bigcap_{k\in K}\{x \in \mathbf{R}_+^n : x^k = 0\}$.

Proof. Since $a^{ii}(\cdot,\cdot) \geq 0$, we have a.s.:

$$\begin{aligned} 0 &\leq \int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} a^{ii}(s,X_s)\,ds \\ &= \int_0^t \mathbf{1}\{X_{s-}^k = 0, \forall k \in K\} a^{ii}(s,X_{s-})\,ds \\ &\leq \int_0^t \mathbf{1}\{X_{s-}^i = 0\} a^{ii}(s,X_{s-})\,ds \\ &= \int_0^t \mathbf{1}\{X_{s-}^i = 0\}\, d[X^i,X^i]_s^c. \end{aligned}$$

But, by [11, Corollary 1, p. 168] we have a.s.:

$$\int_0^t \mathbf{1}\{X_{s-}^i = 0\}\, d[X^i,X^i]_s^c = \int_0^\infty L^i(t,r)\mathbf{1}\{r = 0\}\,dr = 0.$$

Therefore, since $a^{ii}(s, x^{0_K}) > 0$ for a.e. $s \in [0,t]$, we conclude that a.s., $\mathbf{1}\{X_s^k = 0, \forall k \in K\} = 0$ for a.e. $s \in [0,t]$, from where the first part of the lemma follows. As for the second part, note that by the first one and Fubini's theorem:

$$0 = \mathsf{E}\left[\int_0^t a^{ii}(s, X_s)\mathbf{1}\{X_s^k = 0, \forall k \in K\}\, ds\right]$$

$$= \int_0^t \mathsf{E}\left[a^{ii}(s, X_s)\mathbf{1}\{X_s^k = 0, \forall k \in K\}\right] ds$$

and therefore, since $a^{ii}(\cdot,\cdot) \geq 0$, we conclude that for a.e. $s \in [0,t]$:

$$\mathsf{E}\left[a^{ii}(s, X_s)\mathbf{1}\{X_s^k = 0, \forall k \in K\}\right] = 0.$$

The second part of the lemma now follows again by the hypothesis $a^{ii}(s, x^{0_K}) > 0$ for a.e. $s \in [0,t]$. □

Remark 2.2. Note from Lemma 2.1, if for some $t \in \mathbf{R}_+$ and $i \in \{1,\dots,n\}$ we have $a^{ii}(s, x^{0_i}) > 0$ for a.e. $s \in [0,t]$, then P_s^X does not charge the ith face of $\mathbf{R}_+^n$, $\{x \in \mathbf{R}_+^n : x^i = 0\}$, for a.e. $s \in [0,t]$ as well.

Lemma 2.2. *Let $t, r \in \mathbf{R}_+$, $i \in \{1,\dots,n\}$, and ψ be a bounded Borel measurable function from $\mathbf{R}_+^n$ to $\mathbf{R}$. Furthermore, for each $\varepsilon \in \mathbf{R}_+$, let $\mathbf{R}_+^n(i,r,\varepsilon) = \{x \in \mathbf{R}_+^n : r \leq x^i \leq r + \varepsilon\}$. Assume that there exists $\eta \in \mathbf{R}_+$, $\eta > 0$, for which:*

$$\int_0^t \int_{\mathbf{R}_+^n(i,r,\eta)} |\psi(x^{r_i})| a^{ii}(s,x) P_s^X(dx)\, ds < \infty.$$

Then we have:

$$\mathsf{E}\left[\int_0^t \psi(X_s) L^i(ds, r)\right] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \int_{\mathbf{R}_+^n(i,r,\varepsilon)} \psi(x^{r_i}) a^{ii}(s,x) P_s^X(dx)\, ds.$$

Proof. By [11, Corollary 1, p. 168], for each $\varepsilon \in \mathbf{R}_+$ we have a.s.:

$$\int_r^{r+\varepsilon} L^i(\cdot, u)\, du = \int_0^{\cdot} \mathbf{1}\{r \leq X_{s-}^i \leq r + \varepsilon\}\, d[X^i, X^i]_s^c$$

$$= \int_0^{\cdot} \mathbf{1}\{r \leq X_{s-}^i \leq r + \varepsilon\} a^{ii}(s, X_{s-})\, ds$$

$$= \int_0^{\cdot} \mathbf{1}\{r \leq X_s^i \leq r + \varepsilon\} a^{ii}(s, X_s)\, ds.$$

Then, by using Fubini's theorem, for each $\varepsilon \in (0, \eta]$ we have:

$$\frac{1}{\varepsilon} \int_r^{r+\varepsilon} \mathsf{E}\left[\int_0^t \psi(X_s^{r_i}) L^i(ds, u) \right] du$$
$$= \frac{1}{\varepsilon} \int_0^t \mathsf{E}\left[\psi(X_s^{r_i}) \mathbf{1}\{r \le X_s^i \le r + \varepsilon\} a^{ii}(s, X_s) \right] ds$$
$$= \frac{1}{\varepsilon} \int_0^t \int_{\mathrm{R}_+^n(i,r,\varepsilon)} \psi(x^{r_i}) a^{ii}(s, x) P_s^X(dx)\, ds.$$

Now, from [11, Theorem 50, p. 166] we know that, a.s., $L^i(s, r)$ can only increase at times s when $X_s^i = r$, and therefore

$$\mathsf{E}\left[\int_0^t \psi(X_s) L^i(ds, r) \right] = \mathsf{E}\left[\int_0^t \psi(X_s^{r_i}) L^i(ds, r) \right].$$

Since furthermore

$$\mathsf{E}\left[\int_0^t \psi(X_s^{r_i}) L^i(ds, r) \right] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_r^{r+\varepsilon} \mathsf{E}\left[\int_0^t \psi(X_s^{r_i}) L^i(ds, u) \right] du,$$

we conclude

$$\mathsf{E}\left[\int_0^t \psi(X_s) L^i(ds, r) \right] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \int_{\mathrm{R}_+^n(i,r,\varepsilon)} \psi(x^{r_i}) a^{ii}(s, x) P_s^X(dx)\, ds$$

and the lemma is then proved. □

The following corollary, even though not used in the paper, is useful in applications when densities exist.

Corollary 2.1. *Let $t, r \in \mathbf{R}_+$, $i \in \{1, \dots, n\}$, and ψ be a bounded Borel measurable function from $\mathbf{R}_+^n$ to $\mathbf{R}$. Assume that $P_s^X(dx)$ admits a (jointly measurable in s and x) density $p_{X,s}(x)$ (w.r.t. Lebesgue measure), and that there exists $\eta \in \mathbf{R}_+$, $\eta > 0$, for which*

$$\int_0^t \int_{\mathrm{R}_+^{n-1}} |\psi(x^{r_i})| \sup_{r \le x^i \le r+\eta} \{a^{ii}(s, x) p_{X,s}(x)\}\, dx^{\neq i}\, ds < \infty.$$

Furthermore, assume that for a.e. $s \in [0,t]$, $\lim_{x^i \downarrow r}\{a^{ii}(s,x)p_{X,s}(x)\}$ *exists and is finite a.e. over the support of* $\psi(x^{r_i})$ *(as a subset of* $\mathbf{R}_+^{n-1}$*). Then we have:*

$$\mathsf{E}\left[\int_0^t \psi(X_s)L^i(ds,r)\right] = \int_0^t \int_{\mathrm{R}_+^{n-1}} \psi(x^{r_i}) \lim_{x^i \downarrow r}\{a^{ii}(s,x)p_{X,s}(x)\}\, dx^{\neq i}\, ds.$$

Proof. Set $h(s,x^{\neq i}) = |\psi(x^{r_i})| \sup_{r \le x^i \le r+\eta}\{a^{ii}(s,x)p_{X,s}(x)\}$ and

$$g(s) = \int_{\mathrm{R}_+^{n-1}} h(s,x^{\neq i})\, dx^{\neq i},$$

$s \in [0,t]$, $x^{\neq i} \in \mathbf{R}_+^{n-1}$. Since

$$\int_0^t \int_{\mathrm{R}_+^n(i,r,\eta)} |\psi(x^{r_i})| a^{ii}(s,x) P_s^X(dx)\, ds = \int_0^t \int_{\mathrm{R}_+^n(i,r,\eta)} |\psi(x^{r_i})| a^{ii}(s,x) p_{X,s}(x)\, dx\, ds$$
$$\le \eta \int_0^t g(s)\, ds$$

and $\int_0^t g(s)ds < \infty$, by Lemma 2.2 we conclude:

$$\mathsf{E}\left[\int_0^t \psi(X_s)L^i(ds,r)\right] = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \int_{\mathrm{R}_+^n(i,r,\varepsilon)} \psi(x^{r_i}) a^{ii}(s,x) p_{X,s}(x)\, dx\, ds.$$

Now, for a.e. $s \in [0,t]$ we have, $|\psi(x^{r_i})a^{ii}(s,x)p_{X,s}(x)| \le h(s,x^{\neq i})$ for all $x \in \mathbf{R}_+^n(i,r,\eta)$, $\int_{\mathrm{R}_+^{n-1}} h(s,x^{\neq i})\, dx^{\neq i} = g(s) < \infty$, and $\psi(x^{r_i})a^{ii}(s,x)p_{X,s}(x) \to \psi(x^{r_i}) \lim_{x^i \downarrow r}\{a^{ii}(s,x)p_{X,s}(x)\}$, as $x^i \downarrow r$, a.e. over the support of $\psi(x^{r_i})$. Therefore, by Lebesgue's dominated convergence theorem we conclude that, for a.e. $s \in [0,t]$,

$$\lim_{x^i \downarrow r} \int_{\mathrm{R}_+^{n-1}} \psi(x^{r_i}) a^{ii}(s,x) p_{X,s}(x)\, dx^{\neq i} = \int_{\mathrm{R}_+^{n-1}} \psi(x^{r_i}) \lim_{x^i \downarrow r}\{a^{ii}(s,x)p_{X,s}(x)\}\, dx^{\neq i}$$

and hence, for a.e. $s \in [0,t]$ as well,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\mathrm{R}_+^n(i,r,\varepsilon)} \psi(x^{r_i}) a^{ii}(s,x) p_{X,s}(x)\, dx = \int_{\mathrm{R}_+^{n-1}} \psi(x^{r_i}) \lim_{x^i \downarrow r}\{a^{ii}(s,x)p_{X,s}(x)\}\, dx^{\neq i}.$$

Finally, for each $\varepsilon \in (0, \eta]$ and $s \in [0, t]$,

$$\left| \frac{1}{\varepsilon} \int_{\mathrm{R}_+^n(i,r,\varepsilon)} \psi(x^{r_i}) a^{ii}(s,x) p_{X,s}(x)\, dx \right| \leq \frac{1}{\varepsilon} \int_{\mathrm{R}_+^n(i,r,\varepsilon)} |\psi(x^{r_i})| a^{ii}(s,x) p_{X,s}(x)\, dx \leq \frac{1}{\varepsilon}\, \varepsilon g(s) = g(s)$$

and therefore, by using again Lebesgue's dominated convergence theorem ($\int_0^t g(s)\, ds < \infty$), the corollary follows. □

Remark 2.3. Note when a^{ii} is continuous in the space variable (e.g., σ satisfies a local Lipschitz condition), $a^{ii}(s,x)$ can be replaced by $a^{ii}(s,x^{r_i})$ in the conclusion of Corollary 2.1, being then pull out of the limit there, if the corresponding hypothesis on the existence of the limit $\lim_{x^i \downarrow r}\{a^{ii}(s,x) p_{X,s}(x)\}$ is replaced by the respective one on the limit $\lim_{x^i \downarrow r} p_{X,s}(x)$.

Lemma 2.3. *Let $t \in \mathbf{R}_+$ and $i \in K \subseteq \{1, \ldots, n\}$. Assume that there exists $j \in K$, $j \neq i$, such that $a^{jj}(s, x^{0_{K\setminus\{i\}}}) > 0$ for a.e. $s \in [0,t]$. Then we have a.s.:*

$$\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} L^i(ds, 0) = 0.$$

Proof. Since $j \in K \setminus \{i\}$ is such that $a^{jj}(s, x^{0_{K\setminus\{i\}}}) > 0$ for a.e. $s \in [0,t]$, from Lemma 2.1 we conclude that

$$P_s^X\big\{x \in \mathbf{R}_+^n : x^k = 0, \forall k \in K \setminus \{i\}\big\} = 0$$

for a.e. $s \in [0,t]$ as well. Therefore, for each $\eta \in (0, \infty)$ we have:

$$\int_0^t \int_{\mathrm{R}_+^n(i,r,\eta)} \mathbf{1}\big\{x^k = 0, \forall k \in K \setminus \{i\}\big\} a^{ii}(s,x) P_s^X(dx)\, ds = 0$$

and hence, from Lemma 2.2 we conclude:

$$\mathsf{E}\left[\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} L^i(ds, 0)\right] = 0.$$

But, since $\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} L^i(ds, 0) \geq 0$, the lemma follows. □

Remark 2.4. Lemmas 2.1 to 2.3 above do not require the special invertibility structure imposed on R. This special structure is required for Lemmas 2.4 and 2.5 below, as well as for the main results of the paper to be given in the next section.

Lemma 2.4. *Define the* $\mathbf{R}^{n\times n}$*-valued process* $(R_t)_{t\geq 0}$ *as* $R_t^{ij} = R^{ij}\,\mathbf{1}\{X_{t-}^i = 0\}$ *if* $i \neq j$, $R_t^{ii} = R^{ii}$. *Then* R_t *is non-singular for each* $(t,\omega) \in \mathbf{R}_+ \times \Omega$.

Proof. Let $t \in \mathbf{R}_+$, and define the random index set Λ_t as $\{i \in \{1,\dots,n\} : X_{t-}^i > 0\}$. Then it is easy to see that

$$det[R_t] = \begin{cases} det[R], & \text{if } |\Lambda_t| = 0, \\ \prod_{i=1}^{n} R^{ii}, & \text{if } |\Lambda_t| = n, \\ det\big[R^{(\Lambda_t)}\big] \prod_{i\in\Lambda_t} R^{ii}, & \text{if } 0 < |\Lambda_t| < n, \end{cases} \tag{2.3}$$

where $det[\cdot]$ denotes the determinant of the corresponding matrix in $\mathbf{R}^{n\times n}$. Therefore, since by assumption we have that every principal submatrix extracted from R is non-singular, we conclude $det\,[R_t] \neq 0$ for each $(t,\omega) \in \mathbf{R}_+ \times \Omega$, and the lemma is proved. □

Lemma 2.5. *Define the* $\mathbf{R}^{n\times n}$*-valued process* $(r_t)_{t\geq 0}$ *as* $r_t = R_t^{-1}$, *where* $(R_t)_{t\geq 0}$ *is the same as in Lemma 2.4. Then for each* $i \in \{1,\dots,n\}$ *we have a.s.:*

$$Z_\cdot^i = \sum_{j=1}^{n} \int_0^{\cdot} r_s^{ij}\,\mathbf{1}\{X_s^j = 0\}\Big[\frac{1}{2}L^j(ds,0) - b^j(s,X_s)\,ds\Big].$$

Proof. Let $i \in \{1,\dots,n\}$. By applying the Meyer−Itô formula [11, Theorem 51, p. 167] to X^i with convex function $f(x^i) = (x^i)^+ = \max\{0,x^i\}$, $x^i \in \mathbf{R}$, and using the facts that $X^i \geq 0$ and $\mathbf{1}\{X_{s-}^i > 0\} = 1 - \mathbf{1}\{X_{s-}^i = 0\}$, we obtain:

$$\int_{0+}^{\cdot} \mathbf{1}\{X_{s-}^i = 0\}\,dX_s^i = \sum_{0<s\leq\cdot} \mathbf{1}\{X_{s-}^i = 0\}X_s + \frac{1}{2}L^i(\cdot,0).$$

Then, by using equation (2.1) we conclude:

$$\begin{aligned} \frac{1}{2}L^i(\cdot,0) = &\int_0^{\cdot} \mathbf{1}\{X_{s-}^i = 0\}b^i(s,X_{s-})\,ds \\ &+ \sum_{j=1}^{n}\int_0^{\cdot} \mathbf{1}\{X_{s-}^i = 0\}\sigma^{ij}(s,X_{s-})\,dW_s^j + \sum_{j=1}^{n} R^{ij}\int_0^{\cdot} \mathbf{1}\{X_{s-}^i = 0\}\,dZ_s^j \end{aligned}$$

where we have changed 0+ by 0 in all the integrals above since all the integrators are continuous. Now, the continuous local martingale term $G_\cdot^i \triangleq$

$\sum_{j=1}^{n} \int_0^{\cdot} \mathbf{1}\{X^i_{s-} = 0\}\sigma^{ij}(s, X_{s-})\, dW^j_s$ is such that $G^i_0 = 0$ and

$$
\begin{aligned}
[G^i, G^i]_{\cdot} &= \sum_{j,k=1}^{n} \int_0^{\cdot} \mathbf{1}\{X^i_{s-} = 0\}\sigma^{ij}(s, X_{s-})\sigma^{ik}(s, X_{s-})\, d[W^j, W^k]_s \\
&= \int_0^{\cdot} \mathbf{1}\{X^i_{s-} = 0\} a^{ii}(s, X_{s-})\, ds \\
&= \int_0^{\cdot} \mathbf{1}\{X^i_{s-} = 0\}\, d[X^i, X^i]^c_s \\
&= \int_0^{\infty} L^i(\cdot, r)\mathbf{1}\{r = 0\}\, dr = 0
\end{aligned}
$$

where for the fourth equality above we have used [11, Corollary 1, p. 168]. Therefore, we have $G^i \equiv 0$ a.s. Also, since $\int_{\mathrm{R}_+} X^i_s\, dZ^i_s = 0$ and $Z^i_0 = 0$, we have

$$
\int_0^{\cdot} \mathbf{1}\{X^i_{s-} = 0\}\, dZ^i_s = \int_0^{\cdot} \mathbf{1}\{X^i_s = 0\}\, dZ^i_s = Z^i_{\cdot} \quad \text{a.s.}
$$

Moreover, since $L^i(0,0) = 0$ and $L^i(s,0)$ can only increase at times s when $X^i_s = 0$ [11, Theorem 50, p. 166], we also have $L^i(\cdot,0) = \int_0^{\cdot} \mathbf{1}\{X^i_s = 0\} L^i(ds,0)$ a.s. Hence, we may write:

$$
\sum_{j=1,\ j\neq i}^{n} \mathbf{1}\{X^i_{t-} = 0\} R^{ij}\, dZ^j_t + R^{ii}\, dZ^i_t = \mathbf{1}\{X^i_t = 0\}\Big[\frac{1}{2} L^i(dt, 0) - b^i(t, X_t)\, dt\Big].
$$

The lemma now follows by using Lemma 2.4 and the fact that $Z_0 = 0$. □

3. Main results

Using the lemmas given in the previous section, we can now state and prove the main results of the paper.

Theorem 3.1. *Let $t \in \mathbf{R}_+$, and assume that there exist $i, j \in K \subseteq \{1, \ldots, n\}$, $i \neq j$, such that $a^{ii}(s, x^{0_{K\setminus\{j\}}}) > 0$ and $a^{jj}(s, x^{0_{K\setminus\{i\}}}) > 0$ for a.e. $s \in [0,t]$. Then for each $q \in K$ we have a.s.:*

$$
\int_0^t \mathbf{1}\{X^k_s = 0, \forall k \in K\}\, dZ^q_s = 0.
$$

Proof. Let $q \in K$. From Lemma 2.5 we have a.s.:

$$\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\}\, dZ_s^q$$
$$= \sum_{l=1}^n \int_0^t r_s^{ql} \mathbf{1}\big\{X_s^k = 0, \forall k \in K \cup \{l\}\big\} \Big[\frac{1}{2} L^l(ds, 0) - b^l(s, X_s)\, ds\Big].$$

From the assumptions and Lemma 2.1 we conclude, $m\big\{s \in [0,t] : X_s^k = 0, \forall k \in K \cup \{l\}\big\} = 0$ a.s. for each $l \in \{1, \dots, n\}$, and therefore

$$\int_0^t r_s^{ql} \mathbf{1}\big\{X_s^k = 0, \forall k \in K \cup \{l\}\big\} b^l(s, X_s)\, ds = 0$$

also a.s. for each $l \in \{1, \dots, n\}$. Furthermore, from the assumptions and Lemma 2.3 we conclude that for each $l \in \{1, \dots, n\}$ we have a.s., $\mathbf{1}\big\{X_s^k = 0, \forall k \in K \cup \{l\}\big\} = 0$ for a.e. $s \in [0,t]$ with respect to the measure $L^l(ds, 0)$. Hence, we have a.s.:

$$\int_0^t r_s^{ql} \mathbf{1}\big\{X_s^k = 0, \forall k \in K \cup \{l\}\big\} L^l(ds, 0) = 0$$

for each $l \in \{1, \dots, n\}$ as well. The theorem is now proved. □

Theorem 3.2. *Assume that for each $i \in \{1, \dots, n\}$ we have $a^{ii}(t, x^{0_i}) > 0$ for a.e. $t \in \mathbf{R}_+$. Then for each $i \in \{1, \dots, n\}$ we have a.s.:*

$$Z_\cdot^i = \frac{L^i(\cdot, 0)}{2R^{ii}}.$$

Proof. Let $t \in \mathbf{R}_+$ and $i \in \{1, \dots, n\}$. We have a.s.:

$$Z_t^i = \int_0^t \mathbf{1}\{X_s^i = 0\}\, dZ_s^i = \sum_{j=1}^n \int_0^t r_s^{ij} \mathbf{1}\{X_s^i = X_s^j = 0\} \Big[\frac{1}{2} L^j(ds, 0) - b^j(s, X_s)\, ds\Big]$$
$$= \frac{1}{2} \int_0^t r_s^{ii} \mathbf{1}\{X_s^i = 0\} L^i(ds, 0)$$

where the first equality above follows by the same arguments as in the proof of Lemma 2.5, the second one from Lemma 2.5 itself, and the third one from the fact that $a^{ii}(s, x^{0_i}) > 0$ for a.e. $s \in [0, t]$ (by the same arguments as in the proof of Theorem 3.1). Then, since also for each $j \neq i$ we have $a^{jj}(s, x^{0_j}) > 0$ for a.e. $s \in [0, t]$, from Lemma 2.3 we conclude that a.s.:

$$Z_t^i = \frac{1}{2} \int_0^t r_s^{ii} \mathbf{1}\{X_s^i = 0, X_s^j > 0, \forall j \neq i\} L^i(ds, 0)$$
$$= \frac{1}{2} \int_0^t r_s^{ii} \mathbf{1}\{X_{s-}^i = 0, X_{s-}^j > 0, \forall j \neq i\} L^i(ds, 0).$$

But, it is easy to see that on $\{(s, \omega) \in \mathbf{R}_+ \times \Omega : X_{s-}^i(\omega) = 0, X_{s-}^j(\omega) > 0, \forall j \neq i\}$, $r_s^{ii}(\omega) = 1/R^{ii}$. Hence, we have a.s.:

$$Z_t^i = \frac{1}{2R^{ii}} \int_0^t \mathbf{1}\{X_{s-}^i = 0, X_{s-}^j > 0, \forall j \neq i\} L^i(ds, 0)$$
$$= \frac{1}{2R^{ii}} \int_0^t \mathbf{1}\{X_s^i = 0\} L^i(ds, 0) = \frac{L^i(t, 0)}{2R^{ii}}$$

where the second equality above follows again from Lemma 2.3, and the last one by the same arguments as in the proof of Lemma 2.5. Therefore, for each $t \in \mathbf{R}_+$ we have $Z_t^i = L^i(t, 0)/2R^{ii}$ a.s., and indistinguishability now follows from almost surely sample path continuity. The theorem is proved. □

Corollary 3.1. *Assume the same as in Theorem 3.2 above. Then for each $i \in \{1, \ldots, n\}$, a.s., the random measures in t, dZ_t^i and $L^i(dt, 0)$, are supported by the same set in $\mathbf{R}_+$ and, moreover, this set is contained in $\{t \in \mathbf{R}_+ : X_t^i = 0, X_t^j > 0, \forall j \neq i\}$.*

Proof. Follows directly from Lemma 2.3 (or Theorem 3.1) and Theorem 3.2. □

Corollary 3.2. *For each $i \in \{1, \ldots, n\}$, set $V^i = X^i - R^{ii} Z^i$. Then, under the same assumptions as in Theorem 3.2 above, for each $i \in \{1, \ldots, n\}$ we have a.s.:*

$$L^i(\cdot, 0) = 2 \sup_{s \in [0, \cdot]} \max\{-V_s^i, 0\}.$$

Proof. Follows directly from Theorem 3.2 and (2.2). □

4. Case of a hyper-rectangular state space

The results obtained in the previous sections can be readily adapted to the case of a hyper-rectangular state space contained in $\mathbf{R}_+^n$. In that direction, let

$u = (u^1, \dots, u^n) \in \mathbf{R}_+^n$ with each $u^i > 0$, and consider the following reflected diffusion with signed jumps constrained to lie in $\times_{i=1}^n [0, u^i]$:

$$X_t = X_0 + \int_0^t b(s, X_{s-})\, ds + \int_0^t \sigma(s, X_{s-})\, dW_s + \sum_{0<s\leq t} \Delta X_s + RZ_t - \widetilde{R}\widetilde{Z}_t \quad (4.1)$$

where X, b, σ and W are as before, but of course with the state space of X, as well as the spatial domain of b and σ, now replaced by $\times_{i=1}^n [0, u^i]$, the definition of Z remains unchanged, and

- $\widetilde{Z} = (\widetilde{Z}_t)_{t\geq 0} = (\widetilde{Z}_t^1, \dots, \widetilde{Z}_t^n)_{t\geq 0}$ is a continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted, $\mathbf{R}_+^n$-valued process, with each $\widetilde{Z}^i$ non-decreasing, null at zero, and such that $\int_{\mathrm{R}_+} (u^i - X_s^i)\, d\widetilde{Z}_s^i = 0$.

- $R = (R^{ij})_{i,j\in\{1,\dots,n\}}, \widetilde{R} = (\widetilde{R}^{ij})_{i,j\in\{1,\dots,n\}} \in \mathbf{R}^{n\times n}$ are completely-S matrices, satisfying the following additional requirement: let M be the matrix with blocks $M^{11} = R$, $M^{12} = -\widetilde{R}$, $M^{21} = -R$ and $M^{22} = \widetilde{R}$; then, we assume that for each $k \in \{1, \dots, n\}$, every $k \times k$ principal submatrix of M is non-singular, i.e., for each $K \subseteq \{1, \dots, 2n\}$ with $|K| \geq n$, $M^{(K)}$ is invertible, with the same meaning as before for $M^{(K)}$. Note these conditions are satisfied, for example, by real triangular matrices R and $\widetilde{R}$ with strictly positive diagonal elements.

Models as the one in (4.1) appear in the context of queueing theory, when work arrives to a network of finite-buffer queues. There, with X^i representing the buffer content in the ith network element and u^i the maximum buffer allocation allowed in that element, the restriction $0 \leq X \leq u$ comes naturally into play, work arriving in excess of this maximum being lost, see for example [1].

Like in the previous case, we assume the existence of a three-tuple $(X, Z, \widetilde{Z})$, as already described, satisfying (4.1). (Note now the continuous finite variation process in the canonical decomposition of $X - X_0$ is $A_\cdot = \int_0^\cdot b(X_{s-})\, ds + RZ_\cdot - \widetilde{R}\widetilde{Z}_\cdot$, $A_0 = 0$.)

Note that, by writing $X^i = \widetilde{V}^i - \widetilde{R}^{ii}\widetilde{Z}^i$ from equation (4.1), the regulator process $\widetilde{Z}^i$ of X^i at level u^i can be characterized as

$$\widetilde{Z}_\cdot^i = \frac{1}{\widetilde{R}^{ii}} \sup_{s\in[0,\cdot]} \max\{\widetilde{V}_s^i - u^i, 0\}. \quad (4.2)$$

(Obviously, the comments immediately below (2.2) also apply to this case, mutatis-mutandis.)

Also, note that by the jointly continuity in t and right continuity in r of $L^i(t, r)$, we have a.s. $L^i(\cdot, u^i) = 0$. Hence, the results in this section will be established in terms of

$$L^i(t,u^i-) = \lim_{r\uparrow u^i} L^i(t,r) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}\{u^i-\varepsilon \le X_s^i \le u^i\}\, d[X^i,X^i]_s^c$$

where, since $\sum_{0<s\le t}|\Delta X_s| < \infty$ a.s. for each $t>0$, the above limits exist and are equal by [11, Theorem 56 and Corollary 3, p. 176 and 178, resp.]. Note that $L^i(\cdot,u^i-)$ is a.s. continuous, non-decreasing, and can only increase at times t when $X_t^i = u^i$.

The corresponding versions of Lemmas 2.1 and 2.3 are stated in Lemmas 4.1 and 4.2 below, respectively. We omit the proofs since they follow by the same arguments as before. (For the proof of Lemma 4.2, note the version of Lemma 2.2 when the integrator is $L^i(ds,r-)$, instead of $L^i(ds,r)$, is obvious.)

Lemma 4.1. *Let $t\in\mathbf{R}_+$ and $K,\widetilde{K}\subseteq\{1,\dots,n\}$, such that $K\cap\widetilde{K}=\emptyset$. Assume that there exists $i\in K\cup\widetilde{K}$ such that $a^{ii}(s,x^{0_K},x^{u_{\widetilde{K}}})>0$ for a.e. $s\in[0,t]$. Then we have a.s.:*

$$m\big\{s\in[0,t]: X_s^k=0, \forall k\in K, X_s^q=u^q, \forall q\in\widetilde{K}\big\}=0.$$

Also, for a.e. $s\in[0,t]$ we have:

$$P_s^X\big\{x\in\mathbf{R}_+^n : x^k=0, \forall k\in K, x^q=u^q, \forall q\in\widetilde{K}\big\}=0.$$

Lemma 4.2. *Let $t\in\mathbf{R}_+$ and $K,\widetilde{K}\subseteq\{1,\dots,n\}$, such that $K\cap\widetilde{K}=\emptyset$. If $i\in K$ and there exists $j\in K\cup\widetilde{K}$, $j\neq i$, such that $a^{jj}(s,x^{0_{K\setminus\{i\}}},x^{u_{\widetilde{K}}})>0$ for a.e. $s\in[0,t]$, then we have a.s.:*

$$\int_0^t \mathbf{1}\big\{X_s^k=0, \forall k\in K, X_s^q=u^q, \forall q\in\widetilde{K}\big\}L^i(ds,0)=0.$$

In the same way, if $i\in\widetilde{K}$ and there exists $j\in K\cup\widetilde{K}$, $j\neq i$, such that $a^{jj}(s,x^{0_K},x^{u_{\widetilde{K}\setminus\{i\}}})>0$ for a.e. $s\in[0,t]$, then we have a.s.:

$$\int_0^t \mathbf{1}\big\{X_s^k=0, \forall k\in K, X_s^q=u^q, \forall q\in\widetilde{K}\big\}L^i(ds,u^i-)=0.$$

Remark 4.1. Lemmas 4.1 and 4.2 above do not require the special invertibility structure imposed on M (recall M is the matrix with blocks $M^{11}=R$, $M^{12}=-\widetilde{R}$, $M^{21}=-R$ and $M^{22}=\widetilde{R}$). This special structure is required for Lemma 4.3 below, as well as for the subsequent results.

Lemma 4.3. *Define the* $\mathbf{R}^{n\times n}$*-valued processes* $(R_t)_{t\geq 0}$, $(\widetilde{R}_t)_{t\geq 0}$, $(\rho_t)_{t\geq 0}$ *and* $(\widetilde{\rho}_t)_{t\geq 0}$ *as follows:*

$$\begin{aligned} R_t^{ij} &= R^{ij}\mathbf{1}\{X_{t-}^i = 0\} \qquad \text{if } i \neq j, \quad R_t^{ii} = R^{ii}; \\ \widetilde{R}_t^{ij} &= \widetilde{R}^{ij}\mathbf{1}\{X_{t-}^i = u^i\} \qquad \text{if } i \neq j, \quad \widetilde{R}_t^{ii} = \widetilde{R}^{ii}; \\ \rho_t^{ij} &= -R^{ij}\mathbf{1}\{X_{t-}^i = u^i\}; \qquad \text{and} \qquad \widetilde{\rho}_t^{ij} = -\widetilde{R}^{ij}\mathbf{1}\{X_{t-}^i = 0\}. \end{aligned}$$

Furthermore, set $(M_t)_{t\geq 0}$ *as the* $\mathbf{R}^{2n\times 2n}$*-valued process with blocks* $M_t^{11} = R_t$, $M_t^{12} = \widetilde{\rho}_t$, $M_t^{21} = \rho_t$ *and* $M_t^{22} = \widetilde{R}_t$. *Then* M_t *is non-singular for each* $(t,\omega) \in \mathbf{R}_+ \times \Omega$.

Proof. Let $t \in \mathbf{R}_+$, and define the random index sets Λ_t and $\widetilde{\Lambda}_t$ as $\{i \in \{1,\ldots,n\} : X_{t-}^i > 0\}$ and $\{i \in \{n+1,\ldots,2n\} : X_{t-}^{i-n} < u^{i-n}\}$, respectively. Then, it is easy to see that

$$det[M_t] = \begin{cases} det[R]\displaystyle\prod_{i=1}^n \widetilde{R}^{ii} & \text{if } |\Lambda_t| = 0, \\ det\big[\widetilde{R}\big]\displaystyle\prod_{i=1}^n R^{ii} & \text{if } |\widetilde{\Lambda}_t| = 0, \\ \displaystyle\prod_{i=1}^n R^{ii}\widetilde{R}^{ii} & \text{if } \big|\Lambda_t \cup \widetilde{\Lambda}_t\big| = 2n, \\ det\big[M^{(\Lambda_t\cup\widetilde{\Lambda}_t)}\big]\displaystyle\prod_{i\in\Lambda_t} R^{ii} \prod_{j+n\in\widetilde{\Lambda}_t} \widetilde{R}^{jj} & \text{if } |\Lambda_t||\widetilde{\Lambda}_t| > 0 \\ & \text{and } |\Lambda_t \cup \widetilde{\Lambda}_t| < 2n. \end{cases} \tag{4.3}$$

Therefore, since by assumption we have that for each $k \in \{1,\ldots,n\}$, every $k\times k$ principal submatrix extracted from M is non-singular, we conclude $det\,[M_t] \neq 0$ for each $(t,\omega) \in \mathbf{R}_+ \times \Omega$, and the lemma is proved. □

Theorem 4.1. *Assume that for each* $i \in \{1,\ldots,n\}$ *we have* $a^{ii}(t, x^{0_i}) > 0$ *and* $a^{ii}(t, x^{u_i^i}) > 0$ *for a.e.* $t \in \mathbf{R}_+$. *Then for each* $i \in \{1,\ldots,n\}$ *we have a.s.:*

$$Z_\cdot^i = \frac{L^i(\cdot,0)}{2R^{ii}}$$

and

$$\widetilde{Z}_\cdot^i = \frac{L^i(\cdot,u^i-)}{2\widetilde{R}^{ii}}.$$

Proof. Let $i \in \{1,\ldots,n\}$. By applying the Meyer−Itô formula [11, p. 167, Theorem 51] to X^i with convex function $f(x^i) = (x^i)^+ = \max\{0,x^i\}$, $x^i \in \mathbf{R}$, and using equation (4.1), by the same arguments as in the proof of Lemma 2.5 we find:

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{1}\{X_{t-}^i = 0\} R^{ij}\, dZ_t^j + R^{ii}\, dZ_t^i - \sum_{j=1}^{n} \mathbf{1}\{X_{t-}^i = 0\} \widetilde{R}^{ij}\, d\widetilde{Z}_t^j = \frac{1}{2} L^i(dt, 0) \tag{4.4}$$

where we have also used Lemma 4.1 to conclude that, since $a^{ii}(t, x^{0_i}) > 0$ for a.e. $t \in \mathbf{R}_+$, $\int_0^{\cdot} \mathbf{1}\{X_s^i = 0\} b^i(s, X_s)\, ds = 0$ a.s. (note from Lemma 4.1 we have $\int_0^t \mathbf{1}\{X_s^i = 0\} b^i(s, X_s)\, ds = 0$ a.s. for each $t \in \mathbf{R}_+$; indistinguishability follows from continuity). Now, since $L^i(\cdot, u^i) = 0$ a.s., by [11, Corollary 1, p. 177] we have, also a.s.:

$$L^i(\cdot, u^i-) = -2 \int_0^{\cdot} \mathbf{1}\{X_{s-}^i = u^i\} \Big[b^i(s, X_s)\, ds + \sum_{j=1}^{n} R^{ij}\, dZ_s^j - \sum_{j=1}^{n} \widetilde{R}^{ij}\, d\widetilde{Z}_s^j \Big].$$

But, since $a^{ii}(t, x^{u_i^i}) > 0$ for a.e. $t \in \mathbf{R}_+$, again from Lemma 4.1 (by the same arguments as above) the term including the drift is identically zero. Also, since $\int_{\mathrm{R}_+} (u^i - X_s^i)\, d\widetilde{Z}_s^i = 0$ and $\widetilde{Z}_0^i = 0$, we have

$$\int_0^{\cdot} \mathbf{1}\{X_{s-}^i = u^i\}\, d\widetilde{Z}_s^i = \int_0^{\cdot} \mathbf{1}\{X_s^i = u^i\}\, d\widetilde{Z}_s^i = \widetilde{Z}_\cdot^i \quad \text{a.s.}$$

Hence, we may write:

$$-\sum_{j=1}^{n} \mathbf{1}\{X_{t-}^i = u^i\} R^{ij}\, dZ_t^j + \sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{1}\{X_{t-}^i = u^i\} \widetilde{R}^{ij}\, d\widetilde{Z}_t^j + \widetilde{R}^{ii}\, d\widetilde{Z}_t^i = \frac{1}{2} L^i(dt, u^i-). \tag{4.5}$$

Thus, from equations (4.4) and (4.5) ($i = 1, \ldots, n$), Lemmas 4.2 and 4.3, and the fact that $Z_0^i = \widetilde{Z}_0^i = L^i(0,0) = L^i(0, u^i-) = 0$ for each i, we conclude that for each $t \in \mathbf{R}_+$, $Z_t^i = L^i(t,0)/2R^{ii}$ and $\widetilde{Z}_t^i = L^i(t, u^i-)/2\widetilde{R}^{ii}$, a.s., and indistinguishability now follows from almost surely sample path continuity. The theorem is proved. □

Corollary 4.1. *Assume the same as in Theorem* 4.1 *above. Then for each* $i \in \{1, \ldots, n\}$, *a.s., the random measures in* t, dZ_t^i *and* $L^i(dt, 0)$, *are supported by the same set in* $\mathbf{R}_+$ *and, moreover, this set is contained in* $\{t \in \mathbf{R}_+ :\ X_t^i = 0, X_t^j \notin \{0, u^j\}, \forall j \neq i\}$. *This same conclusion holds for the random measures in* t, $d\widetilde{Z}_t^i$ *and* $L^i(dt, u^i-)$, *but now the common support is contained in* $\{t \in \mathbf{R}_+ : X_t^i = u^i, X_t^j \notin \{0, u^j\}, \forall j \neq i\}$.

Proof. Straightforward from Lemma 4.2 and Theorem 4.1. □

The next corollary corresponds to the respective version of Theorem 3.1 in the previous section. For simplicity we will state it under the same assumptions as in Theorem 4.1.

Corollary 4.2. *Let $t \in \mathbf{R}_+$ and $K, \widetilde{K} \subseteq \{1, \ldots, n\}$, such that $K \cap \widetilde{K} = \emptyset$ and $|K \cup \widetilde{K}| \geq 2$. Assume the same as in Theorem 4.1. Then for each $i \in K$ we have a.s.:*

$$\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K, X_s^q = u^q, \forall q \in \widetilde{K}\} dZ_s^i = 0.$$

In the same way, for each $i \in \widetilde{K}$ we have a.s.:

$$\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K, X_s^q = u^q, \forall q \in \widetilde{K}\}\, d\widetilde{Z}_s^i = 0.$$

Proof. Straightforward from Lemma 4.2 and Theorem 4.1. □

The respective version of Corollary 3.2 is obvious from Theorem 4.1 and (4.2).

5. Concluding remarks

In this paper we have established a boundary behavior characterization for reflected diffusions with signed jumps, constrained to lie in the positive orthant of $\mathbf{R}^n$. We have related the regulator processes of such diffusions to their corresponding semi-martingale local times, proving that known properties for SRBMs continue to hold in this more general setting, but requiring, as a trade-off in our proofs, an additional invertibility condition on the reflection matrix. Even though this extra requirement reduces to some extent the scope of our results, we believe it still encompasses a large class of models appearing in applications. Also, signed jumps have been allowed, which is of interest in several areas such as in risk theory or in financial models with claims arising at random times. Finally, we note the results exposed in the paper are not only relevant from the point of view of defining the boundary behavior and obtaining a semi-martingale local time characterization of the regulator processes, but also they are of use in characterizing the stationary distributions of this class of processes, as it was done for example in [10] for the one-dimensional case.

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