Analysis of Load Balancing in Large Heterogeneous Processor Sharing Systems

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Abstract

We analyze randomized dynamic load balancing schemes for multi-server processor sharing systems when the number of servers in the system is large and the servers have heterogeneous service rates. In particular, we focus on the classical power-of-two load balancing scheme and a variant of it in which a newly arrived job is assigned to the server having the least instantaneous Lagrange shadow cost among two randomly chosen servers. The instantaneous Lagrange shadow cost at a server is given by the ratio of the number of unfinished jobs at the server to the capacity of the server. Two different approaches of analysis are presented for each scheme. For exponential job length distribution, the analysis is done using the mean field approach and for more general job length distributions the analysis is carried out assuming an asymptotic independence property. Analytical expressions to compute mean sojourn time of jobs are found for both schemes. Asymptotic insensitivity of the schemes to the type of job length distribution is established. Numerical results are presented to validate the theoretical results and to show that, unlike the homogeneous scenario, the power-of-two type schemes considered in this paper may not always result in better behaviour in terms of the mean sojourn time of jobs.

Keywords: Load balancing, processor sharing, power-of-two, mean field approach, asymptotic independence, insensitivity

Short-title: Power-of-two load balancing for heterogeneous processor sharing systems

1 Introduction

A central problem in a multi-server resource sharing system is to decide which server an incoming job will be routed to. The problem becomes more challenging when the number of servers in the system becomes large. Motivation to consider such systems comes from the study of cloud computing clusters that accommodate a large number of front end servers to process incoming job requests [12]. The choice of the user routing scheme affects the response times for the job completion.

The goal of load balancing techniques is to balance server occupancies in order to reduce overall system delay. In a recent study, both Google and Amazon [17] reported that an increase in average response time of a web search by 500 ms resulted in 1.2% loss of revenue and users. Since delay at a given server increases with the load, a reduction in mean delay can be achieved by reducing the load on the servers.

Centralized hardware load balancers, such as F5 Application Delivery Controller [16], are commonly used in small web server farms that implement a Join-the-Shortest-Queue (JSQ) routing principle to assign jobs to servers. An analysis of the JSQ routing scheme in a system of two parallel servers, each with first-come-first-served (FCFS) service discipline, was done in [10, 11] assuming Poisson arrivals and exponential service time distribution. Optimality of the JSQ scheme, in terms of response time of users, was considered in [7, 19]. It was shown in [19] that, under FCFS service discipline and service distributions having decreasing hazard rate[1], JSQ maximizes the number of jobs that depart from the system in a given amount of time. The JSQ scheme has usually been studied for FCFS servers. Recently Gupta *et al* [9] analyzed the routing scheme under processor sharing service discipline, which is a more common in web server farms. An approximate analysis of the JSQ scheme was presented assuming a finite number of servers in the system and general job length distributions. It was shown that the JSQ scheme, although not optimal in terms of minimizing delay, performs reasonably well in comparison with other schemes of higher complexity.

Unlike small web server farms, a cloud data center accommodates a large number of front end servers for processing job requests. Under such circumstances, a single, central load balancer, capable of continuously monitoring all the servers in the system, is both expensive and wasteful. It is wasteful because a reconfiguration of the load balancer is required each time some servers, having low utilization, are turned off to save consumption of power. These drawbacks of a centralized load balancer have prompted the development of distributed job dispatchers.

With a distributed job dispatcher, JSQ can be implemented by obtaining the number of jobs at each server only at arrival instants of new requests. Although it solves the problem of continuous monitoring of each server, it introduces a new problem of additional delay in the routing of each job. The delay is due to the communication between the job dispatcher and the servers that takes place at the arrival instant of each new job. To avoid the the routing delay caused by the JSQ scheme, randomized routing techniques can be adopted.

Randomized routing can be both static (state independent) and dynamic (state dependent). In state independent routing, a user is routed to a given server with a fixed probability, independent of the loads of the servers in the system. This routing scheme has the advantage that it is simple to implement and incurs no routing delay since no communication is required between the job dispatcher and the servers. An optimal state independent routing scheme for heterogeneous PS server farm was studied in [1] where the weighted mean sojourn time of the users was minimized with respect to the routing probabilities. However, the average response time of jobs under such routing is large due to uneven distribution of loads on the servers.

The mean response time can be reduced significantly by using dynamic state dependent routing

[1]For a service distribution, having cumulative distribution function (CDF) $F(\cdot)$, the hazard rate $h(\cdot)$ is given by $h(t) = f(t)/(1 - F(t))$, where $f(\cdot)$ denotes the probability density function (PDF) of the distribution.

schemes. This is however not always the case in the heterogeneous context. In particular we consider a randomized power-of-two JSQ scheme in the heterogeneous setting for which there are no explicit known results. The scheme can be described as follows. At each arrival instant of a new job request, a set of $d > 1$ servers is chosen uniformly at random from the set of available servers. The new job is then sent to the server having the least number of jobs among the d chosen servers. This routing scheme is referred to as the SQ(d) scheme in the literature [5, 6]. Since d is generally chosen to be a small number, the communication overhead between the job dispatcher and the servers remains small. Classical analyses have only considered the homogeneous (identical server) case with FIFO disciplines. In addition to the SQ(2) scheme we also study a scheme that is related to both the server occupancy and speed that we refer to as the SL(2) scheme which can also be interpreted as the Lagrangian cost for log utility functions [2]. Although this paper focuses on the case of two randomly chosen servers the extension to any finite number of randomly chosen servers (that is even better) is direct at the cost of more notations.

The organization of the paper is as follows. In Section 2 we present the model and provide a description of the load balancing schemes studied in this paper. In Section 3 we present detailed analyses of the three schemes. Section 4 contains numerical results that show that the SL(2) scheme outperforms the other two schemes in terms of average delay and we provide simulation confirmation to demonstrate the *insensitivity* results shown in this paper.

2 System Model

We consider a system consisting of N parallel processor sharing (PS) servers. The servers are indexed by the set $\mathcal{S} = \{1, 2, \ldots, N\}$. The capacity of server $i \in \mathcal{S}$ is denoted by C_i. It is the processing speed experienced by a single job being processed at the server. We refer to the case, where $C_i = C$ for all $i \in \mathcal{S}$, as the *homogeneous* case to distinguish it from the more general case where servers may have different capacities. The general case, referred to as the *heterogeneous* case, is the principal focus of this paper.

Users are assumed to arrive according to a Poisson process with rate $N\lambda$. Each user brings a random amount of work, independent and identically distributed, with a finite mean $\frac{1}{\mu}$. Upon arrival, a user is assigned to one of the N servers according to a load balancing scheme. If the user is assigned to server $i \in \mathcal{S}$, then the rate at which it is served at the server at time t is given by $C_i/x_i(t)$, where $x_i(t)$ denotes the total number of users being processed by server i at time t. The user leaves the system after the completion of its service.

2.1 Optimal state independent load balancing scheme

In this scheme, each user, upon arrival, is assigned to server $i \in \mathcal{S}$ with a fixed probability p_i, independent of the state of the servers in the system. These routing probabilities p_i, $i \in \mathcal{S}$, satisfying $\sum_{i=1}^{N} p_i = 1$ can be chosen in such a way that the average sojourn time of a user in the system is minimized. We shall refer to the load balancing scheme that uses these optimal routing probabilities as the *optimal state independent load balancing scheme*. The implementation of the static load balancing scheme is simple since it does not require the knowledge of the instantaneous states of the servers in the system at the time of an arrival. This scheme, therefore, serves as a benchmark for comparison with the state dependent routing schemes that require the knowledge of the instantaneous states of some servers in the system.

2.2 Dynamic, state dependent load balancing schemes

In state dependent load balancing schemes, job assignment decisions are made based on the instantaneous states of the servers in the system. A well known load balancing scheme known as the join-the-shortest-queue (JSQ) scheme assigns an arriving user to the server having the least number of unfinished jobs. For large server farms (N large) the JSQ scheme requires a lot of overhead to keep track of server occupancies. Therefore, significant additional routing delay might be incurred which adds to the response time of jobs and may result in a reduction in the number of customers and its operator's revenue. In this paper, we consider the well known power-of-two load balancing scheme [15, 18, 5], also known as SQ(2) scheme and a variant of the scheme. It requires the state information of only two servers at each arrival instant and, for the homogeneous scenario, is known to perform exponentially better than the optimal state independent routing scheme. Here we consider the heterogeneous case with M possible server speeds.

2.2.1 Scheme 1: The SQ(2) Scheme

In this scheme, a subset of two servers is selected from the set of N servers uniformly at random at each arrival instant. The arrival is then assigned to the server having the least number of unfinished jobs among the two chosen servers. In case of a tie, the job is assigned to any one of the two servers with equal probability $\frac{1}{2}$. This is the classical *power of two* scheme or SQ(2) scheme [5, 15, 14] applied to the heterogeneous scenario.

2.2.2 Scheme 2: The SL(2) scheme

This is a variant of the original power of two scheme. In this scheme, a random subset of two servers is selected uniformly from the set of N servers at each arrival instant, as is also done in Scheme 1. Let the variable x denote the number of unfinished jobs at a server at the arrival instant and C denote the capacity of the server. The arriving job is assigned to the server having the smallest value of x/C among the two chosen servers. Here, the metric x/C is the instantaneous Lagrange multiplier computed from the social welfare maximization problem of allocating C amount of bandwidth to x users assuming log utility function for each user. Ties are broken in the same way as described for Scheme 1. It is clear that if the two chosen servers have the same number of unfinished jobs then Scheme 2 assigns the incoming job to the server having higher capacity or greater processing speed. As compared to this, for Scheme 1, in the same scenario, the job can be assigned to any one of the two servers with equal probability. Hence, this second scheme favours faster servers.

3 Analysis of the load balancing schemes

In this section, we analyze the load balancing schemes described in the previous section. The analysis of the state independent routing scheme is relatively simple and can be done for any finite value of N. However, the analyses of Scheme 1 and Scheme 2 are done for the situation when $N \to \infty$, i.e., where the number of servers in the system is large.

3.1 Analysis of the optimal state independent load balancing scheme

Let $\vec{p} = \{p_1, p_2, \ldots, p_N\}$ denote the vector of routing probabilities satisfying $\sum_{i=1}^{N} p_i = 1$. Since the input arrival process is Poisson with rate $N\lambda$, the arrival process at each server $i \in \mathcal{S}$ is a Poisson process with rate $p_i N\lambda$ and is independent of the arrival processes at the remaining servers in the system. Therefore, the system can be viewed as a set of independent, parallel M/G/1 processor sharing servers. It is known that, under the condition $\rho_i(\vec{p}) = \frac{p_i N\lambda}{\mu C_i} < 1$, for all $i \in \mathcal{S}$, the system is stable and a unique stationary distribution exists. We call the routing vector $\vec{p}$ as a stable routing vector if this condition holds. The necessary and sufficient condition for existence of stable routing vector is given by the following lemma.

Lemma 3.1 *There exists a stable routing vector $\vec{p}$ if and only if*

$$\frac{N\lambda}{\mu \sum_{i\in\mathcal{S}} C_i} < 1 \tag{3.1}$$

Proof: For a routing vector $\vec{p}$, the traffic intensities, $\rho_i(\vec{p})$, for $i \in \mathcal{S}$, are given by

$$\rho_i(\vec{p}) = \frac{p_i N\lambda}{\mu C_i}.$$

By summing over all i we get

$$\sum_{i\in\mathcal{S}} C_i \rho_i(\vec{p}) = \frac{N\lambda}{\mu} \sum_{i\in\mathcal{S}} p_i = \frac{N\lambda}{\mu}.$$

Therefore, if $\sum_{i\in\mathcal{S}} C_i \le \frac{N\lambda}{\mu}$, then we must have $\rho_i(\vec{p}) \ge 1$ for some i. Thus, (3.1) is a necessary condition for existence of a stable routing vector.

Now, let us assume (3.1) to be true. We form the routing vector $\vec{p}$ as follows:

$$p_i = \frac{C_i}{\sum_{j\in\mathcal{S}} C_j}, \text{ for all } i \in \mathcal{S} \tag{3.2}$$

With this routing vector, the traffic intensities are given by

$$\rho_i(\vec{p}) = \frac{N\lambda}{\mu \sum_{i\in\mathcal{S}} C_i} < 1$$

The inequality in the last expression follows from the hypothesis. Therefore, we see that the routing vector $\vec{p}$ given by (3.2) is stable. Hence, (3.1) is a sufficient condition for existence of a stable routing vector. $\blacksquare$

Let T_i denote the random variable representing the sojourn time of a user at server i. Since each server in the system is a M/G/1 PS server, the expectation of T_i under the stationary distribution is given by $\mathbb{E}[T_i(\vec{p})] = \frac{1}{p_i N\lambda} \frac{\rho_i(\vec{p})}{1-\rho_i(\vec{p})}$.

Note that the above expression remains valid for any distribution of job lengths due to the insensitivity of processor sharing service discipline to the type of service distributions [3]. Hence, the overall mean sojourn time of a job is given by

$$\mathbb{E}[T(\vec{p})] = \sum_{i\in\mathcal{S}} p_i \mathbb{E}[T_i(\vec{p})] = \frac{1}{N\lambda} \sum_{i\in\mathcal{S}} \frac{\rho_i(\vec{p})}{1-\rho_i(\vec{p})} \tag{3.3}$$

We can now formulate the sojourn time minimization problem with respect to the loads as Problem (3.4) below:

$$\begin{aligned} \underset{\vec{\rho}}{\text{Minimize}} \quad & \frac{1}{N\lambda} \sum_{i\in\mathcal{S}} \frac{\rho_i}{1-\rho_i} \\ \text{subject to} \quad & 0 \le \rho_i < 1, \text{for all } i \in \mathcal{S} \\ & \sum_{i\in\mathcal{S}} \rho_i C_i = \frac{N\lambda}{\mu}. \end{aligned} \tag{3.4}$$

We note that, in the above minimization problem, the objective function is strictly convex. Therefore, the problem has a unique solution. The optimal routing vector $\vec{p^*}$ can be calculated from the optimal traffic intensity vector $\vec{\rho^*}$ by using the relation $p_i^* = \frac{\mu}{N\lambda}\rho_i^* C_i$.

To solve the optimization problem given by (3.4) without loss of generality we assume that the servers are ordered as,

$$C_1 \geq C_2 \geq \ldots \geq C_N. \tag{3.5}$$

Let $\mathcal{S}_{\text{opt}} \subset \mathcal{S}$ denote the set of servers chosen under the optimal routing scheme. We have the following result.

Proposition 3.1 *The subset of servers used in the optimal state independent routing scheme is given by $\mathcal{S}_{opt} = (1, 2, \ldots, j^*)$, where j^* is given by*

$$j^* = \sup\left\{ j \leq N : \sum_{i=1}^{j} \sqrt{C_i} > \left(\sum_{i=1}^{j} C_i - \frac{N\lambda}{\mu} \right) \sqrt{\frac{1}{C_j}} \right\}. \tag{3.6}$$

Moreover, the optimal traffic intensities ρ_i^*, for $i \in \mathcal{S}$ satisfy

$$\rho_i^* = \begin{cases} 1 - \sqrt{\frac{1}{C_i}} \frac{\sum_{k \in \mathcal{S}_{opt}} C_k - \frac{N\lambda}{\mu}}{\sum_{k \in \mathcal{S}_{opt}} \sqrt{C_k}}, & \textit{if } i \in \mathcal{S}_{opt} \\ 0, & \textit{otherwise.} \end{cases} \tag{3.7}$$

Proof: The above proposition is special case of Theorem 1 of [1]. ∎

It is easily seen from the proposition and assumption (3.5) that for $i < j$ we have $\rho_i^* \geq \rho_j^*$ and $\mathbb{E}[T_i] \leq \mathbb{E}[T_j]$. From Proposition 3.1 the optimal routing vector $\vec{p^*}$ can be found as

$$p_i^* = \begin{cases} \frac{\mu}{N\lambda} \left(C_i - \sqrt{C_i} \frac{\sum_{k \in \mathcal{S}_{\text{opt}}} C_k - \frac{N\lambda}{\mu}}{\sum_{k \in \mathcal{S}_{\text{opt}}} \sqrt{C_k}} \right), & \text{if } i \in \mathcal{S}_{\text{opt}} \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

The optimal value of the mean sojourn time can be computed using the optimal routing probabilities and (3.3).

3.2 Analysis of the SQ(2) scheme

Unlike the state independent load balancing scheme, in Scheme 1, the job assignment decisions are made based on the loads on the servers in the system at the arrival instants of the jobs. Therefore, the arrival processes at the individual servers are not independent of each other. This makes the exact computation of the stationary distribution extremely difficult for any finite value of N. By using an ansatz based on asymptotic independence or servers, referred to as *propagation of chaos* [8] we can obtain results when $N \to \infty$.

We assume that the set of possible values of capacity that a server can have in the system is finite. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_M\}$ denote the set of possible capacities and N_j denote the number of servers having capacity C_j. Clearly, $\sum_{j=1}^{M} N_j = N$. We assume that $\frac{N_j}{N} \to \gamma_j$ as $N \to \infty$ for each $j \in \{1, 2, \ldots, M\}$. The quantities γ_j, for $j \in \{1, 2, \ldots, M\}$, can be interpreted as the probabilities with which a server, in the infinite system, assumes the different capacity values in $\mathcal{C}$. Hence, we have $\sum_{j=1}^{M} \gamma_j = 1$.

We first present the analysis using mean field approach assuming that the job sizes are exponentially distributed with mean $\frac{1}{\mu}$. We follow the method of analysis outlined in [18, 13, 14]. Let $\mathbf{x}_N(t) = \left\{ x_n^{(j)}(t), 1 \leq j \leq M, n \in \mathbb{Z}_+ \right\}$ denote the state of the system at time t, where $x_n^{(j)}(t) = \frac{1}{N_j} \sum_{n' \geq n} y_{n'}^{(j)}(t)$

and $y_n^{(j)}(t)$ is the number of servers having capacity C_j with exactly n unfinished jobs. Hence, $x_n^{(j)}(t)$ denotes the fraction of servers having capacity C_j with at least n unfinished jobs. Clearly, for any N, the process $\mathbf{x}_N(t)$ is a Markov process on the state space $\prod_{j=1}^M \bar{\mathcal{U}}_{N_j}$, where $\bar{\mathcal{U}}_{N_j}$ is defined as follows:

$$\bar{\mathcal{U}}_{N_j} = \{g = (g_n, n \in \mathbb{Z}_+) : g_0 = 1, g_n \geq g_{n+1} \geq 0, N_j g_n \in \mathbb{N}\ \forall n \in \mathbb{Z}_+\}. \tag{3.9}$$

Let $\mathbf{g} = \left\{g_n^{(j)}, n \in \mathbb{Z}_+, 1 \leq j \leq M\right\}$ be any state in the space $\prod_{j=1}^M \bar{\mathcal{U}}_{N_j}$ and

$$\mathbf{e}(n,j) = \left\{e_k^{(i)}, 1 \leq i \leq M, k \in \mathbb{Z}_+\right\} \in \prod_{j=1}^M \bar{\mathcal{U}}_{N_j} \tag{3.10}$$

be a vector with $e_n^{(j)} = 1$ and $e_k^{(i)} = 0$ for all $i \neq j$, $k \neq n$. It is easy to see that the transition rate from the state $\mathbf{g}$ to the state $\mathbf{g} - \mathbf{e}(n,j)/N_j$, where $n \geq 1$, is given by $\mu C_j N_j \left[g^{(j)}(n) - g^{(j)}(n+1)\right]$. Similarly, the transition rate from state $\mathbf{g}$ to the state $\mathbf{g} + \mathbf{e}(n,j)/N_j$, where $n \geq 1$, is given by $\frac{\lambda}{N}\left[g_{n-1}^{(j)} - g_n^{(j)}\right]\sum_{i=1}^M N_i N_j \left[g_{n-1}^{(i)} + g_n^{(i)}\right]$. For the Markov process $\mathbf{x}_N(t)$, the generator $\mathbf{A}_N$ acting on functions $f : \prod_{j=1}^M \bar{\mathcal{U}}_{N_j} \to \mathbb{R}$ is defined as $\mathbf{A}_N f(\mathbf{g}) = \sum_{\mathbf{h}\neq\mathbf{g}} q_{\mathbf{gh}}\left(f(\mathbf{h}) - f(\mathbf{g})\right)$, where $q_{\mathbf{gh}}$, with $\mathbf{g}, \mathbf{h} \in \prod_{j=1}^M \bar{\mathcal{U}}_{N_j}$, denotes the transition rate from state $\mathbf{g}$ to state $\mathbf{h}$. The generator of the process $\mathbf{x}_N(t)$ is therefore given by

$$\begin{aligned}\mathbf{A}_N f(\mathbf{g}) = \frac{\lambda}{N}\sum_{n\geq 1}\sum_{j=1}^M\sum_{i=1}^M N_i N_j \left[g_{n-1}^{(j)} - g_n^{(j)}\right]\left[g_{n-1}^{(i)} + g_n^{(i)}\right]\left[f(\mathbf{g} + \frac{\mathbf{e}(n,j)}{N_j}) - f(\mathbf{g})\right] \\ + \sum_{n\geq 1}\sum_{j=1}^M \mu C_j N_j \left[g_n^{(j)} - g_{n+1}^{(j)}\right]\left[f(\mathbf{g} - \frac{\mathbf{e}(n,j)}{N_j}) - f(\mathbf{g})\right],\end{aligned} \tag{3.11}$$

The transition semigroup operator $\mathbf{T}_N(t)$, $t \geq 0$, generated by the generator operator $\mathbf{A}_N$ and acting on functions $f : \prod_{j=1}^M \bar{\mathcal{U}}_{N_j} \to \mathbb{R}$ is defined by $\mathbf{T}_N(t) = \exp\left(t\mathbf{A}_N\right) : \prod_{j=1}^M \bar{\mathcal{U}}_{N_j} \to \mathbb{R}$, $t \geq 0$.

Our aim is to derive the stationary behaviour of the process $\mathbf{x}_N(t)$ as $N \to \infty$. To do so, we first define the spaces $\bar{\mathcal{U}}$ and $\mathcal{U}$ as follows

$$\bar{\mathcal{U}} = \{g = (g_n, n \in \mathbb{Z}_+) : g_0 = 1, g_n \geq g_{n+1} \geq 0\ \forall n \in \mathbb{Z}_+\}. \tag{3.12}$$

$$\mathcal{U} = \{g = (g_n, n \in \mathbb{Z}_+) : g_0 = 1, g_n \geq g_{n+1} \geq 0\ \forall n \in \mathbb{Z}_+, \sum_{n=0}^{\infty} g_n < \infty\}. \tag{3.13}$$

We also define the following norm on the spaces $\prod_{j=1}^M \bar{\mathcal{U}}_{N_j}$, $\bar{\mathcal{U}}^M$, and $\mathcal{U}^M$:

$$\|u\| = \sup_{1\leq j\leq M}\sup_{n\in\mathbb{Z}_+}\frac{\left|u_n^{(j)}\right|}{n+1} \tag{3.14}$$

Note that under this norm the space $\bar{\mathcal{U}}^M$ is complete and compact. Through a series of propositions and lemmas we will now establish weak convergence of the Markov process $\mathbf{x}_N(t)$ as $N \to \infty$ to the process $\mathbf{u}(t) = \left\{u_n^{(j)}(t), n \in \mathbb{Z}_+, 1 \leq j \leq M\right\}$, $t \geq 0$, governed by the following system of differential equations:

$$\mathbf{u}(0) = \mathbf{g}, \tag{3.15}$$
$$\dot{\mathbf{u}}(t) = \mathbf{h}(\mathbf{u}(t)), \tag{3.16}$$

where $\mathbf{g} \in \bar{\mathcal{U}}^M$ and for $1 \le j \le M$,

$$h_0^{(j)}(\mathbf{u}) = 0, \tag{3.17}$$
$$h_n^{(j)}(\mathbf{u}) = \lambda \left(u_{n-1}^{(j)} - u_n^{(j)}\right) \sum_{i=1}^{M} \gamma_i \left(u_{n-1}^{(i)} + u_n^{(i)}\right) - \mu C_j \left(u_n^{(j)} - u_{n+1}^{(j)}\right) \tag{3.18}$$

for all $n \ge 1$. We first prove that the above system of differential equations defines a unique process $\mathbf{u}(t, \mathbf{g})$, $t \ge 0$, in the space $\bar{\mathcal{U}}^M$.

Proposition 3.2 *If $\mathbf{g} \in \bar{\mathcal{U}}^M$, then the system (3.15)-(3.18) has a unique solution $\mathbf{u}(t, \mathbf{g})$ in $\bar{\mathcal{U}}^M$ for all $t \ge 0$.*

Proof: Define $\theta(x) = [\min(x, 1)]_+$, where $[z]_+ = \max(0, z)$. Now, we consider the following modification of (3.15)-(3.18).

$$\mathbf{u}(0) = \mathbf{g}, \tag{3.19}$$
$$\dot{\mathbf{u}}(t) = \tilde{\mathbf{h}}(\mathbf{u}(t)), \tag{3.20}$$

where for $1 \le j \le M$,

$$\tilde{h}_0^{(j)}(\mathbf{u}) = 0, \tag{3.21}$$
$$\tilde{h}_n^{(j)}(\mathbf{u}) = \lambda \left[\theta(u_{n-1}^{(j)}) - \theta(u_n^{(j)})\right]_+ \sum_{i=1}^{M} \gamma_i \left[\theta(u_{n-1}^{(i)}) + \theta(u_n^{(i)})\right] - \mu C_j \left[\theta(u_n^{(j)}) - \theta(u_{n+1}^{(j)})\right]_+ \tag{3.22}$$

for all $n \ge 1$. Note that the right hand side of (3.18) and (3.22) are equal if $\mathbf{u} \in \bar{\mathcal{U}}^M$. Therefore, the two systems have the same solution in $\bar{\mathcal{U}}^M$. Also if $\mathbf{g} \in \bar{\mathcal{U}}^M$, then any solution of the modified system remains within $\bar{\mathcal{U}}^M$. This is because of the facts that if $u_n^{(j)}(t) = u_{n+1}^{(j)}(t)$ for some j, n, t, then $h_n^{(j)}(\mathbf{u}(t)) \ge 0$ and $h_{n+1}^{(j)}(\mathbf{u}(t)) \le 0$, and if $u_n^{(j)}(t) = 0$ for some j, n, t, then $h_n^{(j)}(\mathbf{u}) \ge 0$. Hence, to prove the uniqueness of solution of (3.15)-(3.18), we need to show that the modified system (3.19)-(3.22) has a unique solution in $(\mathbb{R}^{\mathbb{Z}_+})^M$.

Using the norm defined in (3.14) and the facts that $|x_+ - y_+| \le |x - y|$ for any $x, y \in \mathbb{R}$, $|a_1 b_1 - a_2 b_2| \le |a_1 - a_2| + |b_1 - b_2|$ for any $a_1, a_2, b_1, b_2 \in [0, 1]$, and $|\theta(x) - \theta(y)| \le |x - y|$ for any $x, y \in \mathbb{R}$ we obtain

$$\|\tilde{\mathbf{h}}(\mathbf{u})\| \le C_1 \tag{3.23}$$
$$\|\tilde{\mathbf{h}}(\mathbf{u_1}) - \tilde{\mathbf{h}}(\mathbf{u_2})\| \le C_2 \|\mathbf{u_1} - \mathbf{u_2}\|, \tag{3.24}$$

where C_1 and C_2 are constants defined as $C_1 = 2\lambda + \mu(\max_{1 \le j \le M} C_j)$ and $C_2 = 8\lambda + 2\mu(\max_{1 \le j \le M} C_j)$. The uniqueness follows from inequalities (3.23) and (3.24) by using Picard's successive approximation technique since $\bar{\mathcal{U}}^M$ is complete under the norm defined in (3.14). ∎

Lemma 3.2 *For each j, n, j', n' and $t \geq 0$, the partial derivatives $\frac{\partial \mathbf{u}(t,\mathbf{g})}{\partial g_n^{(j)}}$, $\frac{\partial^2 \mathbf{u}(t,\mathbf{g})}{\partial {g_n^{(j)}}^2}$, and $\frac{\partial^2 \mathbf{u}(t,\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}}$ exist for $\mathbf{g} \in \bar{\mathcal{U}}^M$ and satisfy*

$$\left| \frac{\partial u_r^{(k)}(t,\mathbf{g})}{\partial g_n^{(j)}} \right| \leq \exp(Bt) \tag{3.25}$$

and

$$\left| \frac{\partial^2 u_r^{(k)}(t,\mathbf{g})}{\partial {g_n^{(j)}}^2} \right|, \left| \frac{\partial^2 u_r^{(k)}(t,\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}} \right| \leq \exp(Bt) + \frac{8\lambda}{B}(\exp(2Bt) - \exp(Bt)), \tag{3.26}$$

where $B = 2(M+2)\lambda + 2\mu \left(\max_{1 \leq j \leq M} C_j\right)$

Proof: Fix j, n, $\mathbf{g}$ and define $\mathbf{u}'(t) = \partial \mathbf{u} / \partial g_n^{(j)}$. If this partial derivative exists, then $\mathbf{u}'(t)$ must satisfy ${u'}_0^{(k)}(t) = 0$, ${u'}_r^{(k)}(0) = \delta_{k,j}\delta_{r,n}$, and (by differentiating (3.18))

$$\frac{d{u'}_r^{(k)}}{dt} = \lambda \left({u'}_{r-1}^{(k)} - {u'}_r^{(k)} \right) \sum_{i=1}^{M} \gamma_i \left(u_{r-1}^{(i)} + u_r^{(i)} \right) + \lambda \left(u_{r-1}^{(k)} - u_r^{(k)} \right) \sum_{i=1}^{M} \gamma_i \left({u'}_{r-1}^{(i)} + {u'}_r^{(i)} \right) - \mu C_k \left({u'}_r^{(k)} - {u'}_{r+1}^{(k)} \right) \tag{3.27}$$

Conversely, if $\mathbf{u}'(t)$ is a solution of the system above, then it must be the required partial derivative. Using Lemma 3.2 of [13] with $a = B$, $b_0 = 0$, and $c = 1$ and the fact that $\left| u_r^{(k)} \right| \leq 1$ for all k,r it can be shown that $\frac{\partial u_r^{(k)}(t,\mathbf{g})}{\partial g_n^{(j)}}$ exists and is bounded as given by (3.25).

Similarly, by differentiating (3.27) again with respect to $g_n^{(j)}$ and $g_{n'}^{(j')}$, we get the systems of equations for $\frac{\partial^2 u_r^{(k)}(t,\mathbf{g})}{\partial {g_n^{(j)}}^2}$ and $\frac{\partial^2 u_r^{(k)}(t,\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}}$, respectively. Lemma 3.1 of [13] can be applied again to these systems to show that the second order partial derivatives also exist and are bounded as given by (3.26). ∎

In the next proposition, we prove that the transition semigroup $\mathbf{T}_N(t)$ of the Markov process $\mathbf{x}_N(t)$ converges as $N \to \infty$ to the transition semigroup of the deterministic process $\mathbf{u}(t,\mathbf{g})$.

Proposition 3.3 *For any continuous function $f : \bar{\mathcal{U}}^M \to \mathbb{R}$ and $t \geq 0$,*

$$\lim_{N \to \infty} \sup_{\mathbf{g} \in \bar{\mathcal{U}}^M} |\mathbf{T}_N(t) f(\mathbf{g}) - f(\mathbf{u}(t,\mathbf{g}))| = 0 \tag{3.28}$$

and the convergence is uniform in t within any bounded interval.

Proof: The argument of this proof is similar to the argument of the proof in [13]. We omit the details and mention the key point that for a function $f : \bar{\mathcal{U}}^M \to \mathbb{R}$ whose derivatives $\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}}$, $\frac{\partial^2 f(\mathbf{g})}{\partial {g_n^{(j)}}^2}$, and $\frac{\partial^2 f(\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}}$ exist for all j, j', n, n', and are uniformly bounded in modulus by some constant, we have

$$N_j \left(f(\mathbf{g} + \frac{\mathbf{e}(n,j)}{N_j}) - f(\mathbf{g}) \right) \to \frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}} \tag{3.29}$$

$$N_j \left(f(\mathbf{g} - \frac{\mathbf{e}(n,j)}{N_j}) - f(\mathbf{g}) \right) \to -\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}} \tag{3.30}$$

uniformly in $\mathbf{g}$ from $\bar{\mathcal{U}}^M$ as $N \to \infty$. ∎

We write $\mathbf{g} \leq \mathbf{g}'$ to mean that $g_n^{(j)} \leq {g'}_n^{(j)}$ hold for all $n \in \mathbb{Z}_+$ and $1 \leq j \leq M$.

Lemma 3.3 *If $\mathbf{g} \leq \mathbf{g}'$, where $\mathbf{g}, \mathbf{g}' \in \bar{\mathcal{U}}^M$, then $\mathbf{u}(t,\mathbf{g}) \leq \mathbf{u}(t,\mathbf{g}')$ for $t \geq 0$.*

Proof: The proof is exactly the same as the proof of Lemma 3.3 of [13] and is therefore omitted. ∎

We now define the quantities $v_n^{(j)}(\mathbf{g}) = \sum_{r \geq n} g_r^{(j)}$, $n \geq 1$, $1 \leq j \leq M$, $\mathbf{g} \in \mathcal{U}^M$. We note that $v_1^{(j)}(\mathbf{g})$ represents the average number of unfinished jobs at the servers having capacity C_j. Since, $\mathbf{g} \in \mathcal{U}^M$, we have $v_1^{(j)}(\mathbf{g}) < \infty$. For a given solution $\mathbf{u}(t,\mathbf{g})$ of the system (3.15)-(3.18), we write $v_n^{(j)}(t,\mathbf{g}) = v_n^{(j)}(\mathbf{u}(t,\mathbf{g}))$. Define the vector $\vec{v}_n(t,\mathbf{g}) = \left(v_n^{(1)}(t,\mathbf{g}), \ldots, v_n^{(M)}(t,\mathbf{g})\right)$.

Lemma 3.4 *If $\mathbf{g} \in \mathcal{U}^M$, then $\mathbf{u}(t,\mathbf{g}) \in \mathcal{U}^M$ for all $t \geq 0$, and*

$$\begin{aligned}\dot{v}_n^{(j)}(t,\mathbf{g}) = \lambda \Bigg[\sum_{i=1}^{M} \gamma_i u_{n-1}^{(j)}(t,\mathbf{g}) u_{n-1}^{(i)}(t,\mathbf{g}) \\ + \sum_{k=n}^{\infty} \sum_{i=1}^{M} \gamma_i \left(u_k^{(i)}(t,\mathbf{g}) u_{k-1}^{(j)}(t,\mathbf{g}) - u_k^{(j)}(t,\mathbf{g}) u_{k-1}^{(i)}(t,\mathbf{g})\right)\Bigg] \\ - \mu C_j u_n^{(j)}(t,\mathbf{g}). \end{aligned} \quad (3.31)$$

Proof: The expression in (3.31) follows directly by summing the right hand side of (3.18) over all $k \geq n$.

From (3.31) it can be shown that $\dot{v}_1^{(j)}(t,\mathbf{g}) \leq 2\lambda$ for all $t \geq 0$. Therefore, if $\mathbf{g} \in \mathcal{U}^M$, then $\mathbf{u}(t,\mathbf{g}) \in \mathcal{U}^M$ for all $t \geq 0$. ∎

Now we derive the stability region of the system under the SQ(2) scheme. Condition (1.2) of [4] derived for more general classes of JSQ networks, is applied to the system under consideration. It is found that for any finite value of N, the system is stable under the Scheme 1 if the following condition is satisfied:

$$\max_{B \subseteq \mathcal{S}} \left\{ \left(\sum_{i \in B} C_i \right)^{-1} \frac{N\lambda}{\mu} \frac{\binom{|B|}{2}}{\binom{N}{2}} \right\} < 1, \quad (3.32)$$

where B is any subset of the set of servers $\mathcal{S}$ and i is the index for the i^{th} server in the system. Therefore, the limiting system is stable if the arrival rate λ satisfies the following condition:

$$\lim_{N \to \infty} \max_{B \subseteq \mathcal{S}} \left\{ \left(\sum_{i \in B} C_i \right)^{-1} \frac{N\lambda}{\mu} \frac{\binom{|B|}{2}}{\binom{N}{2}} \right\} < 1, \quad (3.33)$$

Assuming that the above condition for stability holds, we now derive the form of the stationary tail distribution of number of unfinished jobs at each server for the limiting system.

Proposition 3.4 *Under condition (3.33), the system (3.15)-(3.18) has a unique fixed point $\mathbf{P} \in \mathcal{U}^M$ which satisfies $\mathbf{u}(t,\mathbf{P}) = \mathbf{P}$ for all $t \geq 0$. Further, $\mathbf{P}$ satisfies $P_0^{(j)} = 1$ for all $1 \leq j \leq M$ and*

$$P_{k+1}^{(j)} = \rho_j \left[\gamma_j \left(P_k^{(j)}\right)^2 + P_k^{(j)} \left(\sum_{\substack{i=1 \\ i \neq j}}^{M} \gamma_i P_k^{(i)} \right) + \sum_{\substack{i=1 \\ i \neq j}}^{M} \sum_{l=k}^{\infty} \gamma_i \left(P_{l+1}^{(i)} P_l^{(j)} - P_l^{(i)} P_{l+1}^{(j)} \right) \right] \quad (3.34)$$

for $k = 1, 2, \ldots$ and $1 \le j \le M$. Moreover, under (3.33)

$$\lim_{t\to\infty} \mathbf{u}(t, \mathbf{g}) = \mathbf{P} \text{ for all } \mathbf{g} \in \mathcal{U}^M. \tag{3.35}$$

Thus, there exists a unique probability measure π on $\mathcal{U}^M$ which is invariant under the map $\mathbf{g} \to \mathbf{u}(t, \mathbf{g})$, so that

$$\int f(\mathbf{g}) d\pi(\mathbf{g}) = \int f(\mathbf{u}(t, \mathbf{g})) d\pi(\mathbf{g}) \tag{3.36}$$

for all $t \ge 0$, $f : \bar{\mathcal{U}}^M \to \mathbb{R}$, and $\pi = \delta_{\mathbf{P}}$, the probability measure concentrated on the point $\mathbf{P}$.

Proof: The fixed point $\mathbf{P}$ of the system (3.15)-(3.18) is the solution of the set of equations $\mathbf{h}(\mathbf{P}) = \mathbf{0}$. This system can be equivalently expressed as a fixed point equation given by $\mathbf{u} = \mathbf{\Gamma}(\mathbf{u})$, where $\mathbf{\Gamma} : \mathcal{U}^M \to \mathcal{U}^M$. Equation (3.34) can be obtained by simple manipulations of the system $\mathbf{u} = \mathbf{\Gamma}(\mathbf{u})$. The direct proof of existence and uniqueness of the fixed point $\mathbf{P} \in \mathcal{U}^M$ satisfying (3.34) is difficult since the space $\mathcal{U}^M$ is not compact under the norm considered in this paper and closed form expression for the components of $\mathbf{P}$ cannot be obtained. We defer the proof till the next section where we argue that assuming an asymptotic independence property the limiting tail probabilities found satisfy the same equations as given in (3.34). Therefore, since the limiting system is stable under condition (3.33), the limiting stationary tail distribution must exist and be unique.

The proof of (3.35) follows *mutatis mutandis* from Theorem 1(iii) of [13] using Lemmas 3.3 and 3.4. ∎

Definition 3.1 *A sequence $\{x_n\}_{n\ge1}$ is said to decrease doubly exponentially if and only if there exist positive constants J, $\alpha < 1$, $\beta > 1$ and γ such that $x_n \le \gamma\alpha^{\beta^n}$ for all $n \ge J$.*

Lemma 3.5 *For any fixed j in the set $\{1, 2, \ldots, M\}$, the sequence $\left\{P_n^{(j)}\right\}_n$ decreases doubly exponentially.*

Proof: Since $\mathbf{P} \in \mathcal{U}^M$, $\sum_{k=1}^{\infty} P_k^{(j)} < \infty$ for each j. Hence, $P_k^{(j)} \to 0$ as $k \to \infty$ for each j. Now, from (3.34), it is easy to see that

$$P_{k+1}^{(j)} \le \rho_j \left[\gamma_j \left(P_k^{(j)}\right)^2 + 2P_k^{(j)} \left(\sum_{\substack{i=1\\ i\ne j}}^{M} \gamma_i P_k^{(i)} \right) \right] \tag{3.37}$$

$$\le \rho_j (2 - \gamma_j) \left(\tilde{P}_k\right)^2 \le \delta \tilde{P}_k, \tag{3.38}$$

where $\tilde{P}_k = \max_{1\le j\le M} P_k^{(j)}$ and $\delta = \tilde{P}_k \max_{1\le j\le M} \rho_j(2 - \gamma_j)$. One can choose k sufficiently large such that $\delta < 1$. Hence, we have $\left(\max_{1\le j\le M} P_{k+1}^{(j)}\right) \le \delta \tilde{P}_k$. Similarly we have, $\left(\max_{1\le j\le M} P_{k+n}^{(j)}\right) \le \delta^{2^n-1} \tilde{P}_k$. This proves that the sequence $\left\{P_k^{(j)}\right\}_k$ decreases doubly exponentially for each j. ∎

Proposition 3.5 *Under the condition (3.33), the Markov process $\mathbf{x}_N(t)$ is positive recurrent for all N and hence has a unique invariant distribution π_N for each N. Moreover, $\pi_N \to \delta_P$ weakly as $N \to \infty$, where $\delta_{\mathbf{P}}$ is given by Proposition 3.4, i.e.,*

$$\lim_{N\to\infty} \mathbb{E}_{\pi_N} f(\mathbf{g}) = f(\mathbf{P}) \tag{3.39}$$

for all continuous functions $f : \bar{\mathcal{U}}^M \to \mathbb{R}$.

Proof: The proof of the proposition is similar to proof of Theorem 3 in [13] and hence omitted. ∎

Insensitivity

Now we proceed to analyze Scheme 1 under general job size distributions. It is difficult to analyze the scenario with the method discussed thus far. This is because the process $\mathbf{x}_N(t)$ defined earlier is not a Markov process when the job length distribution is not exponential. To analyze Scheme 1 under general job length distributions, we assume an asymptotic independence property that was proposed as an ansatz in [5, 6] for the homogeneous case. It states that as $N \to \infty$, the individual servers in the system becomes independent of each other and the stationary distribution assumes a product form. More formally, under the stability condition, as $N \to \infty$, the stationary distribution Π^N of the numbers of unfinished jobs at the servers converges to a unique distribution Π on $\mathbb{Z}_+^\infty$ and if $\Pi^{(k)}$ denotes the restriction of Π to the first k coordinates and $\pi = \Pi^{(1)}$ denotes the one dimensional marginal of Π, then for all k,

$$\Pi^{(k)} = \bigotimes_{i=1}^{k} \pi. \tag{3.40}$$

In the homogeneous scenario, this property was proved in [6] for FCFS service discipline and job length distributions having decreasing hazard rate (DHR). The proof uses monotonicity of states and propagation of chaos arguments. Proving the property for processor sharing service discipline and general job length distributions, however, remains an open problem. We assume that the asymptotic independence property holds for the heterogeneous case and derive the form of the stationary distribution of the numbers of unfinished jobs at the servers.

We focus on any particular server (say server 1) in the system. Consider the arrivals that have server 1 as one of its two possible destinations. These arrivals constitute the *potential arrival process*, denoted by A_N^p, at the server. The probability that the server is one among the two choices, picked at the arrival instant of a user, is $\left(1 - \frac{\binom{N-1}{2}}{\binom{N}{2}}\right) = \frac{2}{N}$. Thus, A_N^p is a Poisson process with rate $\frac{2}{N} \times N\lambda = 2\lambda$.

Next, we consider the arrivals that actually join server 1. These constitute the actual arrival process, denoted by A_N^a, at the server. For finite N, this process is not Poisson since a potential arrival to server 1 actually joins the server depending on the number of users present at the other possible destination server. However, as $N \to \infty$, due to the asymptotic independence property stated in (3.40), the numbers of users present at these two servers become independent of each other. As a result, the actual arrival process A_N^a converges to a state dependent Poisson process A^a as $N \to \infty$.

Proposition 3.6 *For Scheme 1, under the condition (3.33), the arrival rate, λ_k, at a particular server when it has k jobs under service is given by*

$$\lambda_k = \lambda \sum_{i=1}^{M} \gamma_i \left(P_k^{(i)} + P_{k+1}^{(i)}\right), \tag{3.41}$$

where $P_k^{(j)}$, for $j \in \{1, 2, \ldots, M\}$ and $k \in \mathbb{Z}_+$, denotes the stationary probability that a server with capacity C_j has atleast k unfinished jobs. Further, we have $P_0^{(j)} = 1$, for all $j \in \{1, 2, \ldots, M\}$ and $P_k^{(j)}$, for $k \in \mathbb{Z}_+$, satisfy (3.34).

Proof: Consider the potential arrivals at a server when the number of users present at the server is k. This arrival actually joins the server either with probability $\frac{1}{2}$ or with probability 1 depending on whether the number unfinished jobs at the other possible destination server is exactly k or greater than

k, respectively. Since a server having capacity C_j is chosen with probability γ_j, the total probability that the potential arrival joins the server at state k is $\sum_{j=1}^{M} \gamma_j \left(0.5\left(P_k^{(j)} - P_{k+1}^{(j)}\right) + P_{k+1}^{(j)}\right) = 0.5\sum_{j=1}^{M} \gamma_j \left(P_k^{(j)} + P_{k+1}^{(j)}\right)$. Therefore, the rate at which arrivals occur at stake k is given by $2\lambda \times 0.5\sum_{j=1}^{M} \gamma_j \left(P_k^{(j)} + P_{k+1}^{(j)}\right)$. This simplifies to (3.41).

Now, from the global balance equation we obtain

$$P_{k+1}^{(j)} - P_{k+2}^{(j)} = \frac{\lambda_k}{\mu C_j}(P_k^{(j)} - P_{k+1}^{(j)}), \text{ for } j \in \{1, 2, \ldots, M\}. \tag{3.42}$$

Substituting the value of λ_k from (3.41) into (3.42) and upon further simplification we get (3.34). ∎

Proof of existence and uniqueness of the fixed point $\mathbf{P}$ *in Proposition 3.4*: Assuming the asymptotic independence amongst any finite set of servers , under the stability condition (3.33), the Markov process is positive recurrent and hence possesses a unique limiting stationary distribution that satisfies (3.34). Furthermore because of stability the mean number of unfinished jobs at any server is finite. Therefore, the solution of (3.34)must lie in the space $\mathcal{U}^M$. From the observation that the equilibrium solution $\mathbf{P}$ of the system (3.15)-(3.18) satisfies (3.34) it follows that the unique fixed point exists in $\mathcal{U}^M$. ∎

Note that the first part of the proof only relies on the asymptotic independence property and does not depend on the assumed distribution of the job lengths. Hence, (3.41) remains valid for any job length distribution. To prove that Proposition 3.6 is valid for any job length distribution, we need to show that the balance equations given by (3.42) hold for any job length distribution. The following result implies that the balance equations indeed hold for any distribution of job lengths.

Lemma 3.6 *For a processor sharing server with state dependent arrival rates* λ_n*, for* $n \in \mathbb{Z}_+$*, and general service distribution* $G(\cdot)$ *with mean* $\frac{1}{\mu}$*, the stationary distribution of the number of users present at the server, if exists, is given by*

$$\pi_n = \pi_0 \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu C}, \tag{3.43}$$

where π_n *denotes the stationary probability that there are* n *users present at the server and* C *denotes the capacity of the server.*

Proof: Let the state of the server be described by the vector $\vec{y} = (y_1, y_2, \ldots, y_n)$, with $y_1 \le y_2 \le \ldots \le y_n$, when there are n users present at the server and y_i is the work performed on the i^{th} user. Clearly, under any job length distribution $G(\cdot)$, this process is a Markov process on a continuous state space $\mathcal{Y} = \{\vec{y} : 0 \le y_1 \le y_2 \ldots \le y_n, n \in \mathbb{Z}_+\}$. Let $T_i\vec{y} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ denote the state of the system immediately after user i leaves the system at state $\vec{y}$ and $(0, \vec{y}) = (0, y_1, y_2, \ldots, y_n)$ denote the state immediately after a user joins the server at state $\vec{y}$. The transition kernel probability density from state $\vec{y}$ to state $T_i\vec{y}$ is given by $\frac{C}{n}\frac{g(y_i)}{\bar{G}(y_i)}$, where $g(\cdot)$ is the probability density function of the job length distribution and $\bar{G}(t) = 1 - G(t)$. Similarly, the transition kernel probability density from state $T_i\vec{y}$ to state $\vec{y}$ is given by $\lambda_{n-1}g(y_i)$. It can be verified that the joint probability density function given by

$$p(\vec{y}) = \frac{n!}{C^n}\left(\prod_{j=0}^{n-1} \lambda_j\right) \pi_0 \prod_{i=1}^{n} \bar{G}(y_i), \text{ for } y \in \mathcal{Y} \tag{3.44}$$

where $\pi_0 = \frac{1}{1+\sum_{n=1}^{\infty}(\prod_{j=0}^{n-1}\lambda_j)}$ satisfies the detailed balance equations given by

$$p(\vec{y})\frac{C}{n}\frac{g(y_i)}{\bar{G}(y_i)} = p(T_i\vec{y})\lambda_{n-1}g(y_i), \tag{3.45}$$

$$p(\vec{y})\lambda_n g(0) = p((0,\vec{y}))\frac{C}{n+1}\frac{g(0)}{\bar{G}(0)}, \tag{3.46}$$

Thus, the density function given by (3.44), satisfies the reversibility condition for continuous space Markov process. Hence, (3.44) gives the stationary probability density of the Markov process. The stationary distribution of the number of users, as given in the statement of the lemma can be derived from (3.44) by integrating $p(\vec{y})$ over all possible vector $\vec{y} \in \mathcal{Y}$ of length n. ∎

Remark 3.1 *The long run probability that a user joins a server with capacity C_j is given by $\frac{N\gamma_j\bar{\lambda}^{(j)}}{N\lambda}$, where, $\bar{\lambda}^{(j)} = \sum_{k=0}^{\infty}\lambda_k\left(P_k^{(j)} - P_{k+1}^{(j)}\right)$ denotes the average arrival rate to a server having capacity C_j. From (3.41) and (3.34), we obtain that $\frac{\gamma_j\bar{\lambda}^{(j)}}{\lambda} = \gamma_j\frac{P_1^{(j)}}{\rho_j}$ for each $j \in \{1,2,\ldots,M\}$. Thus, the long run probability that a user joins a server with capacity C_j is $\gamma_j\frac{P_1^{(j)}}{\rho_j}$.*

Using the stationary distribution derived in Proposition 3.6 and the remark above we first find the mean sojourn time of a user in the system under Scheme 1.

Proposition 3.7 *The mean sojourn time, $\bar{T}$, of a user in the heterogeneous system under Scheme 1 is given by*

$$\bar{T} = \frac{1}{\lambda}\sum_{j=1}^{M}\sum_{k=1}^{\infty}\gamma_j P_k^{(j)}, \tag{3.47}$$

where $P_k^{(j)}$, for $j \in \{1,2,\ldots,M\}$, are as given in Proposition 3.6.

Proof: Let $\bar{T}_j$ denote the mean sojourn time of a user given that it has joined a server having capacity C_j. Now, the expected number of users at a server having capacity C_j is given by $\sum_{k=1}^{\infty}P_k^{(j)}$. Let the average arrival rate at the server be denoted by $\bar{\lambda}^{(j)}$. Thus, applying Little's formula we have $\bar{T}_j = \frac{\sum_{k=1}^{\infty}P_k^{(j)}}{\bar{\lambda}^{(j)}}$

As discussed in the second remark, the long run probability that a user joins a server having capacity C_j is $\frac{\gamma_j\bar{\lambda}^{(j)}}{\lambda}$. Therefore, the overall mean sojourn time is given by $\bar{T} = \sum_{j=1}^{M}\frac{\gamma_j\bar{\lambda}^{(j)}}{\lambda}\bar{T}_j = \frac{1}{\lambda}\sum_{j=1}^{M}\sum_{k=1}^{\infty}\gamma_j P_k^{(j)}$.

3.3 Analysis of the SL(2) scheme

Similar to the SQ(2) scheme, the analysis of the SL(2) is presented assuming $N \to \infty$. For exponential job length distribution, the system can be analyzed using the mean field approach along similar lines as described for the SQ(2) scheme. The limiting deterministic system to which the underlying Markov process converges weakly as $N \to \infty$ is described by $\mathbf{u}(t) = \left\{u_n^{(j)}(t), n \in \mathbb{Z}_+, 1 \le j \le M\right\}$, $t \ge 0$, which satisfies the following system of differential equations:

$$\mathbf{u}(0) = \mathbf{g}, \tag{3.48}$$

$$\dot{\mathbf{u}}(t) = \mathbf{h}(\mathbf{u}(t)), \tag{3.49}$$

where $\mathbf{g} \in \bar{\mathcal{U}}^M$ and for $1 \leq j \leq M$,

$$h_0^{(j)}(\mathbf{u}) = 0, \tag{3.50}$$

$$h_n^{(j)}(\mathbf{u}) = \lambda \left(u_{n-1}^{(j)} - u_n^{(j)}\right) \sum_{i=1}^{M} \gamma_i \left(u_{\{(n-1)C_i/C_j\}}^{(i)} + u_{[(n-1)C_i/C_j]+1}^{(i)}\right) - \mu C_j \left(u_n^{(j)} - u_{n+1}^{(j)}\right) \tag{3.51}$$

for all $n \geq 1$. Here, $[x]$ denotes the greatest integer not exceeding x and $\{x\} = [x] + 1$ if $x \notin \mathbb{Z}_+$ and $\{x\} = x$ if $x \in \mathbb{Z}_+$. The equilibrium point or the fixed point $\mathbf{P}$ of this system can be found by solving the equation $\mathbf{h}(\mathbf{P}) = 0$. The solution $\mathbf{P} \in \mathcal{U}^M$ gives the limiting tail distribution of the numbers of unfinished jobs at the servers.

For general job length distribution, we again assume the ansatz which states that, under condition (3.33), the individual servers in the system becomes independent of each other as $N \to \infty$. Following the same line of arguments as in Scheme 1 we obtain the following main result.

Proposition 3.8 *For Scheme 2, under the condition (3.32), the arrival rate, $\lambda_k^{(j)}$, at a server with capacity C_j when it has k jobs under service is given by*

$$\lambda_k^{(j)} = \lambda \sum_{i=1}^{M} \gamma_i \left(P_{\{kC_i/C_j\}}^{(i)} + P_{[kC_i/C_j]+1}^{(i)}\right), \tag{3.52}$$

where $P_k^{(j)}$, for $j \in \{1, 2, \ldots, M\}$ and $k \in \mathbb{Z}_+$, denotes the stationary probability that a server with capacity C_j has atleast k users under service. Further, we have $P_0^{(j)} = 1$, for all $j \in \{1, 2, \ldots, M\}$ and $P_k^{(j)}$, for $k \in \mathbb{Z}_+$, satisfy

$$P_{k+1}^{(j)} = \rho_j \sum_{l=k}^{\infty} \sum_{i=1}^{M} \gamma_i \left(P_l^{(j)} - P_{l+1}^{(j)}\right) \left(P_{\{lC_i/C_j\}}^{(i)} + P_{[lC_i/C_j]+1}^{(i)}\right) \tag{3.53}$$

Proof: The details are omitted and will appear elsewhere. $\blacksquare$

Note that the limiting stationary distribution does not depend on the type of job length distribution because in the limiting system the Poisson process at a given server is a state dependent Poisson process. Therefore, by Lemma 3.6, Scheme 2 is asymptotically insensitive to job size distributions.

We also note that the state dependent arrival rates at the individual servers in Scheme 2 (given by (3.52)) depend on the capacities of the individual servers while in Scheme 1 (equation (3.41)) they are the same for all the servers. For the servers having the smallest capacity, it is clear by comparing (3.52) and (3.41), that the state dependent arrival rates are smaller in Scheme 2 than in Scheme 1. This is the reason why mean sojourn time of jobs in Scheme 2 is smaller than that in Scheme 1.

The mean sojourn time of a job in the infinite system can be found from the limiting stationary distribution by the application of Little's law. As in Scheme 1, the mean sojourn time $\bar{T}$ of the Scheme 2 is given by $\bar{T} = \frac{1}{\lambda} \sum_{j=1}^{M} \sum_{k=1}^{\infty} \gamma_j P_k^{(j)}$.

4 Numerical results

In this section, we present simulation results to compare the different load balancing schemes considered in this paper. Numerical results, indicating the accuracy of the asymptotic analysis of the SQ(2)

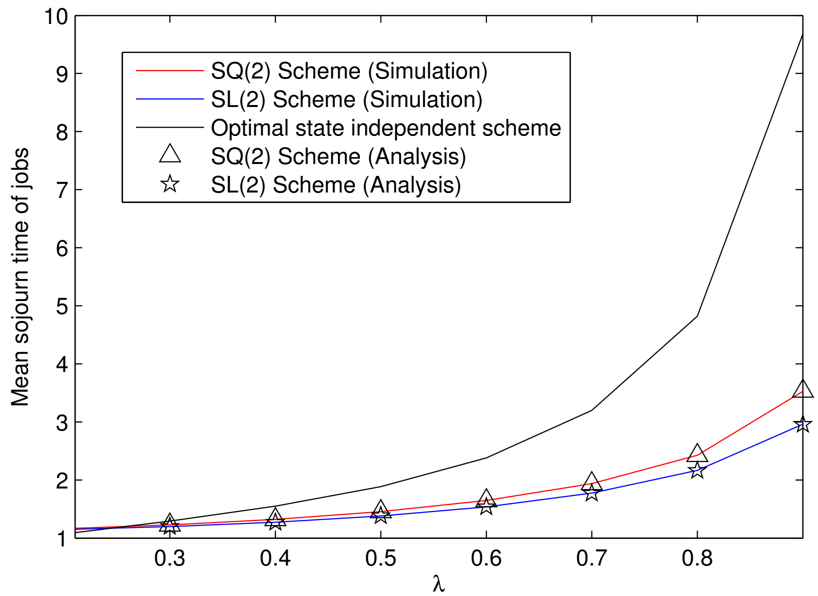

Figure 1: Mean sojourn time jobs as a function of λ for $C_1 = 2/3$, $C_2 = 4/3$, and $\gamma_1 = \gamma_2 = 1/2$

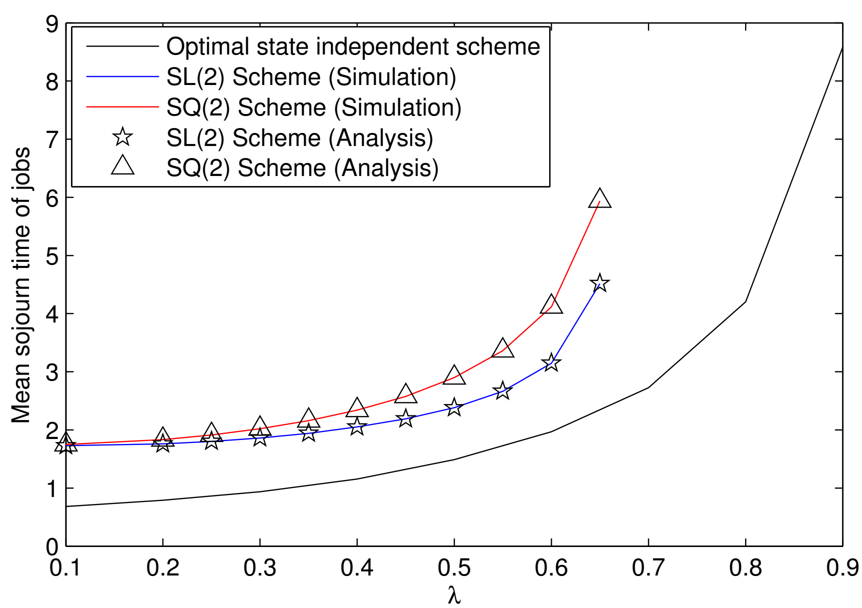

Figure 2: Mean sojourn time jobs as a function of λ for $C_1 = 1/3$, $C_2 = 5/3$, and $\gamma_1 = \gamma_2 = 1/2$

scheme and the SL(2) scheme in predicting their performance in a finite but large system of servers, are also provided. We set $\mu = 1$ in all our simulations.

We simulated the system described in Sec. 2 under two sets of parameter values. In the first, we choose $C_1 = 2/3$, $C_2 = 4/3$, and $\gamma = \gamma_1 = \gamma_2 = \frac{1}{2}$. Using conditions (3.1) and (3.33) it is found that for all the three schemes considered in this paper the stability regions under this parameter setting are identical and is given by $\lambda < 1$. In Figure 1, we plot the average response time of a job in the system as a function of λ for the three schemes under the above mentioned parameter setting. It is observed from the plot that the mean sojourn time of jobs is least under the SL(2) scheme and highest under the optimal state independent load balancing scheme.

However, the performance of the SQ(2) and SL(2) schemes may not always be better than that of the optimal state independent load balancing scheme. To demonstrate this fact we choose the second set of parameter values as follows: $C_1 = 1/3$, $C_2 = 5/3$, and $\gamma_1 = \gamma_2 = 1/2$. Under this parameter setting, the stability region for the optimal state independent scheme is found to be $\lambda < 1$. For the SQ(2) and the SL(2) schemes, however, the stability region is found to be $\lambda < 2/3$, which is a subset of the stability region under the state independent scheme. In Figure 2, we plot the average response time of jobs as a function of λ for the three schemes under the above discussed parameter setting. It is clear that, in this case, the mean response time of jobs is lower for the state independent scheme than that for the SQ(2) and SL(2) schemes.

The reason for this behaviour is as follows: In both SQ(2) and SL(2) schemes, the initial selection of the subset of two servers is done uniformly at random without regard to the server capacities. Therefore, even when the load on the system is high and the slower servers are heavily congested, the SQ(2) and SL(2) schemes select a subset of two slower servers with the same probability with which a subset of two faster servers is selected. This leads to even more congestion at the slower servers and an increase in the overall mean sojourn time of jobs. However, for the optimal state independent load balancing scheme the optimal routing probabilities are functions of the load λ. Therefore, for higher values of λ, most of the traffic is routed to the faster servers.

In Figures 1 and 2 we also observe a good match between the analysis and the simulation results. This implies that the asymptotic results derived for the SQ(2) and SL(2) schemes accurately predict the behaviour of the schemes even when the number of servers in the system is finite but large.

Table 1: Insensitivity of the SQ(2) Scheme

λ	Mean sojourn time $\bar{T}$ Theoretical	Constant Simulation	Power Law ($\beta = 2$) Simulation
0.2	1.1614	1.1623	1.1620
0.3	1.2257	1.2257	1.2261
0.5	1.4547	1.4533	1.3550
0.7	1.9375	1.9377	1.9380
0.8	2.4265	2.4335	2.4330
0.9	3.5300	3.5204	3.5210

Now we investigate the asymptotic insensitivity of the SQ(2) scheme. We present results for two service distributions: i) constant, with mean $\frac{1}{\mu}$, ii) power law distribution whose CDF is given by $F(x) = 1 - x^{-\beta}$ for $x \geq 1$ and $F(x) = 0$, otherwise. The simulation results shown in Table 1 were obtained for $N = 100$ for the SQ(2) scheme. They show that the mean sojourn time remains almost unchanged when the job length distribution type is changed keeping the same value of the mean job size. Thus, we conclude that a sudden change in the type of user flow will not affect the mean sojourn time users in the system as long as the mean of the job length distribution remains the same. This is a great advantage of using processor sharing.

5 Conclusion

In this paper, we considered randomized load balancing schemes for heterogeneous processor sharing systems. We showed that, as in the case of a homogeneous system, the tail distribution of number of customers at a server decreases doubly exponentially for the SQ(2) load balancing scheme. We also analyzed a variant of the SQ(2) scheme that we refer to as SL(2) that takes into account both number of unfinished jobs at a server and its processing speed to make job assignment decisions. This variant is shown to outperform the classical SQ(2) scheme. Both schemes were shown to be asymptotically insensitive to the distribution of job lengths in processor shared systems.

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