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SCALE FUNCTIONS OF LÉVY PROCESSES AND BUSY PERIODS OF FINITE CAPACITY $M/GI/1$ QUEUES

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Abstract

In this paper we use the exit time theory for Lévy processes to derive new closed form results for the busy period distribution of finite capacity fluid $M/G/1$ queues. Based on this result we then obtain the busy period distribution for finite capacity queues with on-off inputs when the off times are exponentially distributed.

Keywords: Queues; busy period; Lévy process; scale function; Laplace transform; on-off inputs

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1. Introduction

In this paper, we derive the busy period characteristics for finite capacity $M/G/1$ queues by exploiting the exit time theory associated with spectrally negative Lévy processes. The model under consideration is the traditional $M/G/1$ model except that the capacity for unfinished work (also known as the workload) is finite, equal to V, and that the workload is "frozen" at level V in the case of overflow. In other words, when

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an overflow occurs due to the arrival of a customer, the amount of work brought by this customer is only partially admitted in the buffer, up to the limit of the free buffer space just before the arrival. This is the main difference with the usual finite capacity $M/G/1/K$ queue where the total amount of work brought by a customer causing an overflow is rejected. These models are also referred to as $M/G/1$ dams. The model considered in this paper is relevant for analyzing partial packet discard techniques in telecommunication networks.

From a theoretical point of view, the workload process in the finite capacity $M/G/1$ queue as described above results in a spectrally positive Lévy process with reflections at the origin and at the buffer limit. The advantage of this approach is that we obtain an explicit characterization of the Laplace transform of the busy period distribution, which to the best of our knowledge, has not been reported in the literature. Based on this characterization we also explicitly obtain the busy period distribution for the case of ON-OFF inputs with off periods being exponentially distributed.

The Laplace transform of the busy period duration in finite capacity $M/M/1$ queues via martingale arguments has been obtained by Kinateder *et al* in [7]. In [8] the authors consider further characteristics related to the time to overflow and the number of overflows during a busy period of finite capacity $M/M/1$ queues. Related martingale based approaches to the excursion analysis of $M/G/1$ and $G/GI/1$ queues with infinite capacity can be found in [5] and [6], respectively.

In this paper, we show how the exit time theory associated with the exit from a domain of spectrally negative Lévy processes can be used to obtain closed form expressions for the Laplace transform of the busy period in finite capacity $M/G/1$ queues. In particular, we use the fact that a scale function can be defined for Lévy processes. This scale function naturally appears when computing various transient characteristics of the spectrally negative Lévy process. The existence of the scale function has been established by Emery in [4] (see also [2] for an exhaustive treatment of Lévy processes). The scale function has recently been used by Avram *et al* [1] in the context of finance in connection with American and Canadian options.

The organization of this paper is as follows: In Section 2, some basic results on spectrally negative Lévy processes are recalled and the scale function together with related functions are introduced. In Section 3, we apply the basic results on

Lévy processes to a finite capacity fluid $M/G/1$ queue and we derive an explicit representation for the Laplace transform of the busy period; the results are then applied to some special cases. The paper concludes with an application of the results to the case of fluid queues with on-off inputs.

2. Lévy processes and preliminary results

Consider a Lévy process with negative jumps $\{X_t\}$ and positive drift. Such a process is also referred to as spectrally negative Lévy process in the technical literature. For all $\theta \geq 0$, the moment generating function $\mathbb{E}[e^{\theta X_t}]$ exists and is such that

$$\mathbb{E}[e^{\theta X_t}] = e^{t\psi(\theta)},$$

for some function ψ, referred to as Lévy exponent. The function ψ is defined for $\Re(\theta) \geq 0$ and its restriction to the non negative real line is strictly convex and is such that $\lim_{\theta\to\infty} \psi(\theta) = \infty$. In the following, we set for $c \in \Re$

$$\psi_c(\theta) = \psi(c+\theta) - \psi(c). \tag{2.1}$$

We now introduce the tool, which will be central in the following (see for instance Bertoin [3]). Define for $q \geq 0$, $\phi_c(q)$ the largest real root of the equation $\psi_c(\theta) = q$, which exists by the strict convexity of the function ψ and is non negative.

Definition 1. (*Scale function.*) For $q \geq 0$, there exists a unique continuous function $W^{(q)} : [0,\infty) \to [0,\infty)$, called the *q-scale function*, such that

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \phi(q) \stackrel{def}{=} \phi_0(q). \tag{2.2}$$

In connection with the function $W^{(q)}$, let us also define the function $Z^{(q)}(x)$, called the *adjoint q-scale function*, as follows.

Definition 2. For $q \geq 0$, let $Z^{(q)} : \mathbb{R} \to [1,\infty)$ be the function defined by

$$Z^{(q)}(x) = 1 + q\int_{-\infty}^x W^{(q)}(z)dz. \tag{2.3}$$

For fixed x, the functions $W^{(q)}(x)$ and $Z^{(q)}(x)$ in variable q may be analytically continued to the whole of the complex plane (see [3] for details). Moreover, it is easily checked that for $\theta > \phi(q)$,

$$\int_0^\infty e^{-\theta x} Z^{(q)}(x)dx = \frac{\psi(\theta)}{\theta(\psi(\theta)-q)}.$$

Define for $a < b$ the hitting times

$$\begin{aligned} T_a^- &= \inf\{t > 0 : X_t < a\}, \\ T_b^+ &= \inf\{t > 0 : X_t > b\}. \end{aligned}$$

Then, we have the following result, which illustrates the importance of the scale function for computing the Laplace transform of hitting times of the Lévy process $\{X_t\}$. See [3, 1].

Proposition 1. *Let $x \in (a,b)$. Conditionally on $X_0 = x$, the Laplace transforms of T_b^+ and T_a^- are given by: for $q \geq 0$,*

$$\mathbb{E}_x\left[e^{-qT_b^+}\mathbb{I}_{\{T_b^+ < T_a^-\}}\right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}, \tag{2.4}$$

$$\mathbb{E}_x\left[e^{-qT_a^-}\mathbb{I}_{\{T_a^- < T_b^+\}}\right] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}. \tag{2.5}$$

Finally, let $W_c^{(q)}$ be the scale function associated with the Lévy process with exponent $\psi_c(\theta)$ and let $Z_c^{(q)}$ be the corresponding function defined by equation (2.3), where $W^{(q)}$ is replaced with $W_c^{(q)}$. The function $Z_v^{(q)}(x)$ is precisely defined by

$$Z_v^{(q)}(x) = 1 + q\int_{-\infty}^x W_v^{(q)}(z)dz. \tag{2.6}$$

for any $q \geq 0$. Then, we have the following important result due to Emery [4].

Proposition 2. *The Laplace transform of the couple $(T_a^-, X_{T_a^-})$, with the initial condition $X_0 = x > a$ is given by: for $u \geq 0$ and v such that $\psi(v) < \infty$,*

$$\mathbb{E}_x\left[e^{-uT_a^- + vX_{T_a^-}}\right] = e^{ux}\left[Z_v^{(p)}(x-a) - \frac{W_v^{(p)}(x-a)p}{\phi_v(p)}\right],$$

where $p = u - \psi(v)$, $W^{(u)}(x) = e^{vx}W_v^{(u-\psi(v))}(x)$ and the function $Z_v^{(q)}(x)$ is defined by equation (2.6).

The above results are used in the next sections to compute different transient characteristics of a fluid reservoir with finite capacity and fed with compound Poisson inputs as well as some characteristics of an $M/G/1$ queue. It is worth noting that the relevance of doubly reflected spectrally negative Lévy processes in queueing theory has already been investigated in the technical literature (see for instance Pistorius [9]).

3. Application to finite capacity fluid $M/G/1$ queues

Consider a buffer with finite capacity V and fed with fluid inputs arriving according to a Poisson with intensity λ; the drain rate from the buffer is taken equal to unity. The ith input brings a random amount of fluid equal to ξ_i into the buffer and we assume that the random variables ξ_i are independent and identically distributed, with general distribution F. In the following, we assume that the distribution F has a Laplace transform F^* defined for $\Re(\theta) \geq 0$ by

$$F^*(\theta) = \int_0^\infty e^{-\theta x} F(dx).$$

Finally, let $\{\tilde{X}_t\}$ be the process defined by:

$$\tilde{X}_t = \tilde{X}_0 + \sum_{i=1}^{A_t} \xi_i - t,$$

where $\{A_t\}$ is a Poisson process with intensity λ.

We assume that a busy period starts at time 0 and we are interested in the busy period duration τ, formally defined by

$$\tau = \inf\{s > 0 : Z_s = 0\},$$

where $\{Z_t\}$ is the process describing the amount of fluid in the system at time t, given by

$$Z_t = \tilde{X}_t - \max\left(0, \sup_{0<s<t} \tilde{X}_s - V\right).$$

$\{Z_t\}$ corresponds to the process $\{\tilde{X}_t\}$ reflected on the boundaries $x = 0$ and $x = V$. Note that $\tilde{X}_0$ has distribution F on $[0, V)$ and has a mass at point V with probability $1 - F(V)$.

The process $\{-\tilde{X}_t \equiv X_t\}$ is a spectrally negative Lévy process. By using the results on Lévy processes recalled in the previous section, we can prove the following result.

Proposition 3. *The Laplace transform of the busy period duration* τ *of the finite capacity* $M/G/1$ *queue is given for* $q \geq 0$ *by*

$$\mathbb{E}\left[e^{-q\tau}\right] = \frac{1}{Z^{(q)}(V)} \int_0^V Z^{(q)}(V-x)F(dx) + (1-F(V))\frac{1}{Z^{(q)}(V)}, \tag{3.1}$$

where $Z^{(q)}$ *is the function associated with the scale function of the spectrally negative process* $\{X_t\}$ *according to equation* (2.3).

Proof. Assume that $\tilde{X}_0 = x$ and define $\tilde{T}_{0,V} = \inf\{s > 0 : \tilde{X}_s \notin (0, V]\}$. Further define the hitting times for the process $\{\tilde{X}_t\}$:

$$\tilde{T}_0^- = \inf\{t > 0 : \tilde{X}_t \leq 0\}, \tag{3.2}$$

$$\tilde{T}_V^+ = \inf\{t > 0 : \tilde{X}_t > V\}. \tag{3.3}$$

We clearly have

$$\tau = \tilde{T}_0^- \mathbb{1}_{\{\tilde{T}_0^- < \tilde{T}_V^+\}} + (\tilde{T}_V^+ + \tau \circ \Theta(V))\mathbb{1}_{\{\tilde{T}_V^+ < \tilde{T}_0^-\}} \tag{3.4}$$

where $\Theta(V)$ denotes the shift operator for which $(\tilde{X} \circ \Theta(V))_0 = V$. By taking into account the memoryless property of the exponential distribution, the Laplace transform of τ conditioned on $\tilde{X}_0 = x$ satisfies

$$\mathbb{E}_x\left[e^{-q\tau}\right] = \mathbb{E}_x\left[e^{-q\tilde{T}_0^-}\mathbb{1}_{\{\tilde{T}_0^- < \tilde{T}_V^+\}}\right] + \mathbb{E}_x\left[e^{-q\tilde{T}_V^+}\mathbb{1}_{\{\tilde{T}_V^+ < \tilde{T}_0^-\}}\right]\mathbb{E}_V\left[e^{-q\tau}\right]. \tag{3.5}$$

To compute the intermediate Laplace transforms appearing in (3.5), we use the results of Section 2 for the spectrally negative process $\{X_t\}$. The Lévy exponent of this process is

$$\psi(\theta) = \theta - \lambda + \lambda \int_0^\infty e^{-\theta x}F(dx). \tag{3.6}$$

Observe that

$$\mathbb{E}_x\left[e^{-q\tilde{T}_0^-}\mathbf{I}_{\{\tilde{T}_0^- < \tilde{T}_V^+\}}\right] = \mathbb{E}_{-x}\left[e^{-qT_0^+}\mathbf{I}_{\{T_0^+ < T_{-V}^-\}}\right], \tag{3.7}$$

$$\mathbb{E}_x\left[e^{-q\tilde{T}_V^+}\mathbf{I}_{\{\tilde{T}_V^+ < \tilde{T}_0^-\}}\right] = \mathbb{E}_{-x}\left[e^{-qT_{-V}^-}\mathbf{I}_{\{T_{-V}^- < T_0^+\}}\right]. \tag{3.8}$$

By using (2.4), (2.5), (3.7) and (3.8), we have

$$\mathbb{E}_x\left[e^{-q\tilde{T}_0^-}\mathbf{I}_{\{\tilde{T}_0^- < \tilde{T}_V^+\}}\right] = \frac{W^{(q)}(V-x)}{W^{(q)}(V)}, \tag{3.9}$$

$$\mathbb{E}_x\left[e^{-q\tilde{T}_V^+}\mathbf{1}_{\{\tilde{T}_V^+ < \tilde{T}_0^-\}}\right] = Z^{(q)}(V-x) - W^{(q)}(V-x)\frac{Z^{(q)}(V)}{W^{(q)}(V)}, \tag{3.10}$$

where $W^{(q)}$ is the scale function of the process $\{X_t\}$ and $Z^{(q)}$ is the function defined by equation (2.3). Hence, we have from equations (3.5), (3.9) and (3.10)

$$\mathbb{E}_x\left[e^{-q\tau}\right] = \frac{W^{(q)}(V-x)}{W^{(q)}(V)} + \left[Z^{(q)}(V-x) - W^{(q)}(V-x)\frac{Z^{(q)}(V)}{W^{(q)}(V)}\right]\mathbb{E}_V\left[e^{-q\tau}\right]. \tag{3.11}$$

By the *Initial Value Theorem*, we have

$$\lim_{\theta\to+\infty}\theta\int_0^\infty e^{-\theta x}W^{(q)}(x)dx = W^{(q)}(0)$$

From Definition 2.2 of the scale function and the definition of ψ, we have $W^{(q)}(0) = 1$. Moreover, from the definition of the function $Z^{(q)}$, we have $Z^{(q)}(0) = 1$. Hence, from (3.11), we obtain

$$\mathbb{E}_V\left[e^{-q\tau}\right] = \frac{1}{W^{(q)}(V)} + \left[1 - \frac{Z^{(q)}(V)}{W^{(q)}(V)}\right]\mathbb{E}_V\left[e^{-q\tau}\right],$$

which entails

$$\mathbb{E}_V\left[e^{-q\tau}\right] = \frac{1}{Z^{(q)}(V)}$$

and then

$$\mathbb{E}_x\left[e^{-q\tau}\right] = \frac{Z^{(q)}(V-x)}{Z^{(q)}(V)}. \tag{3.12}$$

Equation (3.1) is obtained by deconditioning upon x.

In the proof given above, we have used the fact that $W^{(q)}(0) = 1$, which is intimately related to the fact that the Lévy process under consideration is of bounded variation and that the drift of the process is equal to 1. Unfortunately, this situation does not hold for a Lévy process of unbounded variation. However, it has been proved by Pistorius [10] that the basic identity

$$\mathbb{E}\left[e^{-q\sigma_a} \mid X_0 - \underline{X}_0 = x\right] = \frac{Z^{(q)}(x)}{Z^{(q)}(a)},$$

holds for a general Lévy process X, with a running infimum $\underline{X}_t = \inf_{0\le s\le t}(X_s \wedge 0)$ where $\sigma_a = \inf\{t \ge 0 : X_t - \underline{X}_t > a\}$. This identity is similar to equation (3.12). This indicates that the result of Proposition 3 can be extended to general Lévy processes.

In addition, we have so far assumed that the Lévy measure takes the form $\lambda F(dx)$, where F is a probability distribution with a well defined Laplace transform. But, from a theoretical point of view, we may consider a general Lévy process of bounded variation with Lévy measure $\Pi(dx)$ such that $\int_0^\infty (1 \wedge x)\Pi(dx) < \infty$ without guaranteeing that $\int_0^\infty \Pi(dx) < \infty$. In that case, we have a system, where inputs arrive according to a Poisson process with rate λ and in a small interval dt, inputs which bring an amount of work lying in $(x, x + dx)$ arrive with rate $\lambda\Pi(dx)$. If $\int_0^\infty \Pi(dx) = \infty$, the measure Π is no more a probability measure as in the $M/G/1$ queue but an unbounded Radon measure. We thus obtain a generalization of the $M/G/1$ queue, that we denote, for short, by $M/\Pi/1$. Such a system can be seen as a storage model fed with objects, which are such that those of small size arrive a high rate. The noteworthy point is that even for this more complicated system, the result of Proposition 3 still pertains.

Finally, before proceeding to the analysis of some special cases, let us note that when $V \to \infty$ and the load ρ of the queue, defined by

$$\rho = \lambda \int_0^\infty xF(dx),$$

is less than one, reflections at level V rarely occur in a busy period. The reflection condition at level V thus becomes moot and the busy period duration τ should be close to that of the stable infinite capacity $M/G/1$ queue. For Laplace transforms, this means that we should have $\mathbb{E}[e^{-q\tau}] \sim B^*(q)$, where $B^*(q)$ is the Laplace transform of the busy period of the infinite capacity $M/G/1$ queue with mean input rate λ and service time distribution F. This is readily verified by noting that $\phi(q)$ is the pole with the largest real part of the function $Z^{(q)}$ as shown by the following lemma.

Lemma 1. *The non negative real number $\phi(q)$ is the solution with the greatest real part to the equation $\psi(\theta) = q$.*

Proof. The result is proved by using standard techniques, in particular Rouché's theorem: If the functions $f(z)$ and $g(z)$ of the complex argument z are analytic inside a closed contour C and if also $|f(z)| < |g(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C. We precisely show that unique solution of the equation $\psi(z) = \psi(\phi(q))$ in the domain $\{z : \Re(z) \geq \phi(q)\}$ is $\phi(q)$.

We consider the functions

$$g(z) = z - \phi(q) \quad \text{and} \quad f(z) = \lambda \int_0^\infty \left(e^{-zx} - e^{-\phi(q)x}\right) F(dx).$$

Let $R > 0$ and $\varepsilon > 0$. Consider the closed contour C composed of the semi-circles

$\{\phi(q)+Re^{i\theta}:\theta\in[-\pi/2,\pi/2]\}$ and $\{\phi(q)+\varepsilon e^{i\theta}:\theta\in[-\pi/2,\pi/2]\}$, and the segments $[\phi(q)+i\varepsilon,\phi(q)+iR]$ and $[\phi(q)-iR,\phi(q)-i\varepsilon]$ of the imaginary axis $\Re(z)=\phi(q)$.

For sufficiently large R, $|f(z)|<|g(z)|$ when $z=\phi(q)+Re^{i\theta}$ with $\theta\in[-\pi/2,\pi/2]$. When $z=\phi(q)+yi$ with $y\in[-R,\varepsilon]\cup[\varepsilon,R]$,

$$|f(z)|\leq\lambda\int_0^\infty e^{-\phi(q)x}\left|e^{-ixy}-1\right|F(dx)\leq\lambda\int_0^\infty e^{-\phi(q)x}2\left|\sin\left(\frac{yx}{2}\right)\right|F(dx)\leq\rho|y|,$$

where we have used the fact that $|\sin(x)|\leq|x|$ and $\phi(q)\geq 0$ in the last step. Since $\rho<1$, we have $|f(z)|<\sqrt{\phi(q)^2+y^2}=|g(z)|$.

When $z=\phi(q)+\varepsilon e^{i\theta}$ with $\theta\in[-\pi/2,\pi/2]$, we have for small ε

$$f(z)=-\lambda(z-\phi(q))\int_0^\infty xe^{-\phi(q)x}F(dx)+o(\varepsilon)$$

and then $|f(z)|\leq\rho\varepsilon+o(\varepsilon)$. It follows that for sufficiently small ε, $|f(z)|<\varepsilon=|g(z)|$. As a consequence, for all $z\in C$, $|f(z)|<|g(z)|$ when R is sufficiently large and ε is sufficiently small. Since $g(z)$ has no zeros inside C, we deduce that the unique solution of the equation $\psi(z)=\psi(\phi(q))$ in $\{z:\Re(z)\geq\phi(q)\}$ is $\phi(q)$.

By using the above lemma, we deduce that

$$Z^{(q)}(x)\sim-\frac{q}{\phi(q)\psi'(\phi(q))}e^{\phi(q)x}$$

when $x\to\infty$. From equation (3.1), it is easily checked, by using $\int_0^\infty F(dx)=1$, that when $V\to\infty$,

$$\mathbb{E}\left[e^{-q\tau}\right]\to F^*(\phi(q)),$$

where F^* is the Laplace transform of the service time distribution F. Since $\psi(\theta)=\theta-\lambda+\lambda F^*(\theta)$ and $\psi(\phi(q))=q$, we deduce that $F^*(\phi(q))$ is the (unique) root with module less than 1 to the equation $z=F^*(q+\lambda-\lambda z)$, which characterizes the Laplace transform of the busy period of the $M/G/1$ queue.

To conclude this section, let us consider two cases, namely when service times are exponentially distributed and constant.

Corollary 1. *Assume that service times are exponentially distributed with mean $1/\mu$ and that a busy period starts from level $x\in[0,V]$. Then, the Laplace transform of the busy period τ of the reservoir with finite capacity V is given by*

$$\mathbb{E}_x\left[e^{-q\tau}\right]=\frac{(q-\theta_-(q))e^{(V-x)\theta_+(q)}-(q-\theta_+(q))e^{(V-x)\theta_-(q)}}{(q-\theta_-(q))e^{V\theta_+(q)}-(q-\theta_+(q))e^{V\theta_-(q)}},\tag{3.13}$$

where

$$\theta_{\pm}(q) = \frac{-\mu+\lambda+q \pm \sqrt{(-\mu+\lambda+q)^2+4\mu q}}{2}. \tag{3.14}$$

Proof. In the case of exponential service times with mean $1/\mu$, the Lévy exponent of the spectrally negative process $\{-X_t\}$ is given by

$$\psi(\theta) = \theta - \lambda + \frac{\lambda\mu}{(\mu+\theta)}.$$

The function $Z^{(q)}$ is such that

$$\int_0^\infty e^{-\theta x} Z^{(q)}(x)dx = \frac{\theta+\mu-\lambda}{\theta^2-\theta(\lambda+q-\mu)-q\mu}.$$

By partial fraction decomposition, we can write the right hand side of the above equation as

$$\frac{1}{(\theta_+(q)-\theta_-(q))}\left(\frac{\theta_+(q)+\mu-\lambda}{\theta-\theta_+(q)} - \frac{\theta_-(q)+\mu-\lambda}{\theta-\theta_-(q)}\right),$$

where $\theta_{\pm}(q)$ is defined by equation (3.14). Via Laplace inversion, we obtain

$$Z^{(q)}(x) = \frac{1}{\theta_+(q)-\theta_-(q)}\left[(\theta_+(q)+\mu-\lambda)e^{x\theta_+(q)} - (\theta_-(q)+\mu-\lambda)e^{x\theta_-(q)}\right].$$

Thus, from equation (3.12), we have

$$\mathbb{E}_x\left[e^{-q\tau}\right] = \frac{(\theta_+(q)+\mu-\lambda)e^{(V-x)\theta_+(q)} - (\theta_-(q)+\mu-\lambda)e^{(V-x)\theta_-(q)}}{(\theta_+(q)+\mu-\lambda)e^{V\theta_+(q)} - (\theta_-(q)+\mu-\lambda)e^{V\theta_-(q)}}.$$

Observing that $\theta_+(q)+\theta_-(q) = \lambda+q-\mu$ and $\theta_+(q)\theta_-(q) = -q\mu$, equation (3.13) follows.

Remark 1. By replacing μ by μ^{-1} and denoting $F_i = -\theta_i$, equation (3.13) gives the central result in [7] (see Theorem 1).

Let us finally consider the deterministic case, that is, when the amount of fluid brought into the system by inputs is constant equal to some $d > 0$.

Corollary 2. *In the case when inputs are constant and equal to* d*, the Laplace transform of the busy period of the fluid reservoir with finite capacity* V *is given by equation* (3.1)*, where the function* $Z^{(q)}(x)$ *is given by*

$$Z^{(q)}(x) = 1 + q\sum_{n=0}^{\infty}\int_0^{(x-nd)^+} \frac{(-\lambda t)^n}{n!} e^{(\lambda+q)t} dt \tag{3.15}$$

with the notation $(x-a)^+ = \max\{0, x-a\}$.

Proof. In the deterministic input case, the Lévy exponent is given by

$$\psi(\theta) = \theta - \lambda + \lambda e^{-d\theta}$$

and the function $Z^{(q)}$ is such that

$$\int_0^\infty e^{-\theta x} Z^{(q)}(x) dx = \frac{\theta - \lambda + \lambda^{-d\theta}}{\theta(\theta - \lambda + \lambda e^{-d\theta} - q)} = \frac{1}{\theta} + \frac{q}{\theta(\theta - \lambda - q + \lambda^{-d\theta})}.$$

We have the power series expansion

$$\frac{q}{\theta - \lambda - q + \lambda^{-d\theta}} = -q \sum_{n=0}^{\infty} \frac{\lambda^n}{(\lambda + q - \theta)^{n+1}} e^{-nd\theta}$$

The function $\theta \to 1/(\lambda + q - \theta)^{n+1}$ is the Laplace transform of the function

$$f_n : x \to \frac{(-1)^{n+1} x^n}{n!} e^{(\lambda+q)x}.$$

It follows that the term $e^{-nd\theta}/(\lambda+q-\theta)^{n+1}$ is the Laplace transform of the convolution $f_n * \delta_{nd}$, where δ_{nd} is the Dirac mass at point nd. This convolution is given by

$$f_n * \delta_{nd}(x) = \int_0^x f_n(t) \delta_{nd}(x - t) dt = (-1)^{n+1} \frac{[(x - nd)^+]^n}{n!} e^{(\lambda+q)(x-nd)}.$$

Since the Laplace inverse of the function $\theta \to 1/\theta$ is the unit step function, the function $\theta \to e^{-nd\theta}/(\theta(\lambda + q - \theta)^{n+1})$ is the Laplace transform of the function

$$x \to (-1)^{n+1} \int_0^x \frac{[(t - nd)^+]^n}{n!} e^{(\lambda+q)(t-nd)} dt.$$

The function $Z^{(q)}(x)$ is then given by

$$Z^{(q)}(x) = 1 + q \sum_{n=0}^{\infty} \int_0^x \frac{(-\lambda(t - nd)^+)^n}{n!} e^{(\lambda+q)(t-nd)} dt,$$

which completes the proof of equation (3.15).

To conclude this section, note that if $d \geq V$ then we deduce from equation (3.15) that the Laplace transform of the busy period is equal to

$$\mathbb{E}[e^{-q\tau}] = \frac{\lambda + q}{\lambda + q e^{(\lambda+q)V}},$$

since

$$Z^{(q)}(V) = \frac{\lambda}{\lambda + q} + \frac{q}{\lambda + q} e^{(\lambda+q)V}.$$

This relation readily follows from the fact that

$$\tau \stackrel{d}{=} V \mathbb{1}_{\{E_\lambda > V\}} + (\tau + E_\lambda) \mathbb{1}_{\{E_\lambda < V\}},$$

where E_λ is an exponential random variable independent of τ.

4. Application to queues with fluid on-off inputs

We investigate in this section how results for the $M/G/1$ queue can be exploited to study finite capacity buffers with fluid On-Off inputs when the Off periods are exponentially distributed. Consider a fluid buffer with finite capacity V, fed with an On-Off type arrival process and drained at constant rate c. The On period has a general distribution G while the distribution of Off periods is exponential. During the On period the fluid arrives at a constant rate h.

Let us consider a busy period with length τ^f in this fluid queue for some initial level x. Consider an $M/G/1$ queue with constrained workload V, with inter-arrival times having the same distribution as the Off periods in the fluid model, and the service time distribution F related to G as

$$F(v) = G\left(\frac{v}{h-c}\right).$$

Let τ be the busy period in this $M/G/1$ queue and $\{\tilde{X}_t\}$ be defined as before for this queue. From the analysis in the previous section we have the Laplace transform of τ conditioned on $\tilde{X}_0 = x$. Then, the busy period in the original fluid buffer is given by τ plus the sum of length of the On periods in this busy period. Observe that the sum of the On periods is $\frac{c\tau - x}{h-c}$ plus the length of the time the workload stays at level V during the busy period. To calculate this period we shall look at the joint distribution of T^-_{-V} and $X_{T^-_{-V}}$ for the process $\{X_t\}$. This is because $\inf\{t > 0 : \tilde{X}_t \geq V\} \equiv \inf\{t > 0 : -X_t \geq V\} \equiv \inf\{t > 0 : X_t \leq -V\}$. The Lévy process $\{X_t\}$ has the exponent

$$\psi(\theta) = c\theta - \lambda \int_0^\infty (1 - e^{-\theta x}) dG(x) = c\theta - \lambda \int_0^\infty (1 - e^{-\theta(h-c)x}) dF(x).$$

We have the following result (see [3]).

Lemma 2. *For $u \geq 0$ and v such that $\psi(v) < \infty$ the joint Laplace transform of T^-_{-V} and $X_{T^-_{-V}}$ is given by (with $X_0 = -x$):*

$$\mathbb{E}_{-x}\left[e^{-uT^-_{-V} + vX_{T^-_{-V}}} \mathbb{1}_{\{T^-_{-V} < T^+_0\}}\right] = e^{-vx}\left[Z_v^{(p)}(V-x) - W_v^{(p)}(V-x)\frac{Z_v^{(p)}(V)}{W_v^{(p)}(V)}\right], \quad (4.1)$$

where $p = u - \psi(v)$

Observe that τ^f in terms of the stopping times $\tilde{T}_0^-$ and $\tilde{T}_V^+$ of the process $\{\tilde{X}_t\}$ can be written as

$$\tau^f = \left(\frac{c\tilde{T}_0^- - x}{h-c}\mathbb{1}_{\{\tilde{T}_0^- < \tilde{T}_V^+\}}\right) + \left[\left(\frac{c\tilde{T}_V^+ - x}{h-c} + \frac{X_{\tilde{T}_V^+} - V}{h-c} + \tau^f \circ \Theta(V)\right)\mathbb{1}_{\{\tilde{T}_V^+ < \tilde{T}_0^-\}}\right], \quad (4.2)$$

where Θ is the shift operator defined as in equation (3.4). In equation (4.2), the quantity $(X_{\tilde{T}_V^+} - V)/(h-c)$ is the fraction of time when the buffer level stays at level V during the On period in which the buffer level reaches V. By using equation (4.2), we can prove the following result.

Proposition 4. *The Laplace transform of τ^f conditioned on $\tilde{X}_0 = x$ can be expressed as:*

$$\mathbb{E}_x\left[e^{-q\tau^f}\right] = e^{\frac{qx}{h-c}}\frac{W^{(q_1)}(V-x)}{W^{(q_1)}(V)} + \frac{e^{\frac{qV}{h-c}}\left[Z^{(p_1)}_{\frac{q}{h-c}}(V-x) - e^{\frac{qx}{h-c}}W^{(q_1)}(V-x)\frac{Z^{(p_1)}_{\frac{q}{h-c}}(V)}{W^{(q_1)}(V)}\right]}{e^{\frac{qV}{h-c}}Z^{(p_1)}_{\frac{q}{h-c}}(V) - c(1-e^{-\frac{qV}{h-c}})W^{(q_1)}(V)}, \quad (4.3)$$

where, $q_1 = \frac{qc}{h-c}$ and $p_1 = q_1 - \psi(\frac{q}{h-c})$.

Proof. From equation (4.2), the Laplace transform of τ^f conditioned on $\tilde{X}_0 = x$ satisfies

$$\mathbb{E}_x\left[e^{-q\tau_f}\right] = e^{\frac{qx}{h-c}}\mathbb{E}_x\left[e^{-\frac{qc}{h-c}\tilde{T}_0^-}\mathbb{1}_{\{\tilde{T}_0^- < \tilde{T}_V^+\}}\right] + e^{\frac{q(x+V)}{h-c}}\mathbb{E}_x\left[e^{-\frac{qc}{h-c}\tilde{T}_V^+ - \frac{q}{h-c}X_{\tilde{T}_V^+}}\mathbb{1}_{\{\tilde{T}_V^+ < \tilde{T}_0^-\}}\right]\mathbb{E}_V\left[e^{-q\tau^f}\right]. \quad (4.4)$$

The two conditional expectations on the right hand side of (4.4) can be written in terms of the hitting times for the spectrally negative process $\{-X_t\}$ as

$$\mathbb{E}_x\left[e^{-q\tau_f}\right] = e^{\frac{qx}{h-c}}\mathbb{E}_{-x}\left[e^{-\frac{qc}{h-c}\tilde{T}_0^+}\mathbb{1}_{\{\tilde{T}_0^+ < \tilde{T}_{-V}^-\}}\right] + e^{\frac{q(x+V)}{h-c}}\mathbb{E}_{-x}\left[e^{-\frac{qc}{h-c}\tilde{T}_{-V}^- + \frac{q}{h-c}X_{\tilde{T}_{-V}^-}}\mathbb{1}_{\{\tilde{T}_{-V}^- < \tilde{T}_0^+\}}\right]\mathbb{E}_V\left[e^{-q\tau^f}\right]. \quad (4.5)$$

Substituting the expressions for the expectations in (4.5) from (2.4) and (4.1), we get

$$\mathbb{E}_x\left[e^{-q\tau^f}\right] = e^{\frac{qx}{h-c}}\frac{W^{(q_1)}(V-x)}{W^{(q_1)}(V)} + e^{\frac{qV}{h-c}}\left[Z^{(p_1)}_{\frac{q}{h-c}}(V-x) - W^{(p_1)}_{\frac{q}{h-c}}(V-x)\frac{Z^{(p_1)}_{\frac{q}{h-c}}(V)}{W^{(p_1)}_{\frac{q}{h-c}}(V)}\right]\mathbb{E}_V\left[e^{-q\tau^f}\right]. \quad (4.6)$$

From [1, Remark 3], we have $W^{(q_1)}(x) = e^{\frac{qx}{h-c}}W^{(p_1)}_{\frac{q}{h-c}}(x)$. From equation (4.6), we obtain

$$\mathbb{E}_x\left[e^{-q\tau^f}\right] = e^{\frac{qx}{h-c}}\frac{W^{(q_1)}(V-x)}{W^{(q_1)}(V)} + e^{\frac{qV}{h-c}}\left[Z^{(p_1)}_{\frac{q}{h-c}}(V-x) - e^{\frac{qx}{h-c}}W^{(q_1)}(V-x)\frac{Z^{(p_1)}_{\frac{q}{h-c}}(V)}{W^{(q_1)}(V)}\right]\mathbb{E}_V\left[e^{-q\tau^f}\right]. \quad (4.7)$$

By definition (see equation (2.6)), we have $Z^{(p_1)}_{\frac{q}{h-c}}(0) = 1$ and $W^{(q)}(0) = 1/c$. Thus, from equation (4.7), we have

$$\mathbb{E}_V\left[e^{-q\tau^f}\right] = e^{\frac{qV}{h-c}}\frac{1}{cW^{(q_1)}(V)} + e^{\frac{qV}{h-c}}\left[1 - e^{\frac{qV}{h-c}}\frac{Z^{(p_1)}_{\frac{q}{h-c}}(V)}{cW^{(q_1)}(V)}\right]\mathbb{E}_V\left[e^{-q\tau^f}\right],$$

which implies

$$\mathbb{E}_V\left[e^{-q\tau^f}\right] = \frac{1}{e^{\frac{qV}{h-c}}Z^{(p_1)}_{\frac{q}{h-c}}(V) - c(1-e^{-\frac{qV}{h-c}})W^{(q_1)}(V)} \quad (4.8)$$

From equations (4.7) and (4.8), we obtain (4.3).

To conclude this section, note that the quantity $h(X_{\tilde{T}_V^+} - V)/(h-c)$ appearing in equation (4.2) is the amount of fluid lost during an overflow period. Computations similar to those carried out for the derivation of the Laplace transform of the busy period duration τ^f could be performed in order to establish a closed expression for the Laplace transform of the total amount of fluid lost in a busy period.

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