Chapter 3
Physical Layer: Digital Transmission Fundamentals
**Objective**

Given the characteristics of the communication channel and system resources, what is required to achieve the required performance?

- **Performance**: probability of transmission error (bit error rate)

- **Resource**: signal power (energy), bandwidth
Digital Representation of Information

Stream: ex. Voice
A continuous-time varying signal

Two steps: sampling, quantizing
Digital Representation of Information

Stream: ex. Voice

**Sampling:**

Highest frequency $W = 4k$ Hz:

Min. sample rate: $2W = 8$ kHz
Why Digital Communication?

The resistance to signal distortion

- Distortion: unexpected change of signal during transmission due to imperfect channel
- Analog: cannot resist distortion, degraded performance, quality
- Digital: can correct the signal as long as the distortion is not severe enough to change the signal polarity (e.g., from 0 to 1)

Many others: suitable for long-distance transmission, handling many types of services, various protocol (error correction, data encryption, etc.),...
Block diagram of digital transmissions

- eliminate redundancy in the data, send same information in fewer bits
- compression ratio: \#bits (original file) / \#bits (compressed file)
- **lossless** compression: Zip (in windows)
- **lossy** compression: JPEG (compression ratio of about 15), MPEG (e.g., 300Mbps to 6Mbps)
Transforming signals to improve communications performance by increasing the robustness against channel impairments (noise, interference, fading, ...)

- add redundancy: map binary source output sequences of k into binary channel input sequences of n (n>k)
Block diagram of digital transmissions

- Converts the electrical signals (from the source) into a form that is suitable for transmission over the physical media
  - DSL modem, Cable modem, wireless modulator
Block diagram of digital transmissions

source (digital) → source encoder → channel encoder → modulator

add redundancy

frequency moving

internet

modulator → source decoder → channel decoder → demodulator

destination

Video/audio/text → decompression

repair the impaired signal

frequency moving
Layer 1: Physical Layer

- **Service**: transmitting raw bits (0/1) over wire/physical link between two systems

Transmitter:
- Converts information into signal/waveform suitable for transmission

Receiver:
- Converts received signal into form suitable for delivery to user
Communication Channel

- Twisted pair, Coaxial cable, Radio, Light in optical fiber, etc.
- Wireless communication: propagates through the medium as electromagnetic waves

Twisted pair, Coaxial cable, and Electromagnetic waves are shown in the diagram.
Low-pass and Bandpass Channels

- Low-pass channel: low frequency signal can pass the channel while high frequency signal is blocked.
- Bandpass channel: pass the signal with a frequency range \([f_1, f_2]\).
Attenuation

- Attention of a signal is defined as the reduction or loss in signal power as it is transferred over a channel
  - Cause: signal energy absorbed by media or object
  - Computed in dB as

\[
\text{attenuation (dB)} = 10 \log_{10} \frac{P_{in}}{P_{out}}
\]
Channel Capacity and Noise

- The channel capacity is the maximum rate at which bits can be **reliably** transmitted over a channel.

- **Channel noise** affects the reliability:
  - Thermal electronic noise (due to the vibrations of electrons) is inevitable.
Physical Layer

- **Physical Layer**: Implements a digital communication link that delivers bits. Suppose that:

  \[ S_1(t) \quad \text{Send “1”} \quad S_2(t) \quad \text{Send “0”} \]

  The duration of \( S_i(t) \) is \( T \), where \( T \) is called the symbol duration and \( 1/T \) is called the symbol rate/data rate.

- **Objective**: Given the characteristics of the communication channel, what system resources are needed in order to achieve the required performance?
  - **Performance**: probability of transmission error (bit error)
  - **Resource**: signal power (energy), bandwidth

Figure: The Stochastic Process
Additive White Gaussian Noise (AWGN)

- **AWGN Channel**

  \[ \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \]

  \[ h(t) = \delta(t) \iff H(f) = \mathcal{F}[h(t)] = 1 \]

  \[ r(t) = S_i(t) \ast \delta(t) + n(t) = S_i(t) + n(t) \]

- The channel has infinite bandwidth.
Signal + Noise Ratio

High SNR

Low SNR

signal + noise

virtually error-free signal + noise

error-prone
Bandwidth Limited Channel

- Bandwidth Limited Channel

\[ r(t) = S_i(t) * h(t) + n(t) = \int_0^\infty S(t - \tau)h(\tau) \, d\tau + n(t) \]

- Inter-symbol Interference (ISI):
  In band-limited channels, the channel introduces inter-symbol interference, since the transmitted “signal” pulse is “stretched”.

- Question: How to remove the ISI effect?
Random Signal

Random Process

- A random process is a random variable indexed by $t$.
- A random process is a rule for assigning a real-valued time function (or signal) to each outcome $S_i$ of an experiment $E$. For example, take the transmission of two bits:

```
00  X(t,S_1)
01  X(t,S_2)
10  X(t,S_3)
11  X(t,S_4)
```

Transmission of Two Bits
Random Signal

- A random process $x(t,s)$ is
  - A family of deterministic functions, where $t$ and $S$ are variables
  - A random variable at $t=t_0$, $X(t_0,S)$, where $S$ is a variable
  - A single time function given that $S$ is fixed where $t$ is a time variable: $X(t,S_2) = X_2(t)$
  - A real number if both $t$ and $S$ are fixed

- Covariance and correlation functions are:

  \[
  \text{Cov}(X,Y) = E \left[ (X - \mu_X)(Y - \mu_Y) \right] = E[XY] - \mu_X \mu_Y
  \]

  \[
  R_X(t_1,t_2) = E \left[ X(t_1)X(t_2) \right]
  \]
Model of Physical Layer Communications

- **Model**

![Block Diagram of Physical Layer](image)

- For a random process \(X(t)\), with the correlation function defined above, if the correlation function satisfies the following conditions, then the random process is called **Wide Sense Stationary (WSS)**.

\[
E \left[ X(t) X(t - \tau) \right] = R_X(\tau)
\]

\[
E \left[ X(t) \right] = \text{constant}
\]
Model of Physical Layer Communications

Properties of the Autocorrelation Functions

- \( R_X(\tau) = R_X(-\tau) \), \( R_X(\tau) \) is an even function.
  
  **Proof:**
  
  \[
  R_X(\tau) = R_X(t, t - \tau) = E[X(t)X(t - \tau)] \\
  = E[X(t - \tau)X(t)] = R_X(t - \tau, t) = R_X(-\tau)
  \]

- \( |R_X(\tau)| \leq R_X(0) \)
  
  **Proof:**
  
  \[
  E \left[ (X(t) \pm X(t - \tau))^2 \right] \geq 0 \\
  \Rightarrow E[X^2(t) + X^2(t - \tau) \pm 2X(t)X(t - \tau)] \geq 0 \\
  \Rightarrow R_X(0) + R_X(0) \pm 2R_X(\tau) \geq 0 \\
  \Rightarrow \pm R_X(\tau) \leq R_X(0) \Rightarrow |R_X(\tau)| \leq R_X(0)
  \]


Properties of the Autocorrelation Functions (Cont.)

- $R_X(0)$ is the average power of the random process.

Proof:

The average power of $X(t)$ is

$$P = E \left[ \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} X^2(t) dt \right] \geq 0$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} E[X^2(t)] dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(0) dt = R_X(0)$$
Model of Physical Layer Communications

- Wiener-Khintchine (W-K) Relation

\[ R_X(\tau) = \mathcal{S}^{-1} \left[ S_X(f) \right] = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df \]

\[ S_X(f) = \mathcal{S} \left[ R_X(\tau) \right] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau \]

The autocorrelation function and the power spectral density are a Fourier Transform pair.
White Gaussian Noise

- **Conditions for White Gaussian Noise:**
  - \( R_N(\tau) = \frac{N_0}{2} \delta(\tau) \)
  - \( E[n(t)] = 0 \)
  - \( n(t_i) = \text{a Gaussian Random Variable} \)

- **Properties**
  - \( S_n(f) = \Im \left[ R_n(\tau) \right] = \frac{N_0}{2} \)
  - \( P_n = \int_{-\infty}^{\infty} S_n(f) \, df \to \infty \)
  - \( S_{n_0}(f) = |H_d(f)|^2 S_n(f) \)

\( n(t) \xrightarrow{h_d(t)} H_d(f) \xrightarrow{} n_o(t) \)
White Gaussian Noise

Properties

\[ S_{n_0}(f) = \left| H_d(f) \right|^2 S_n(f) \]

\[ n(t) \rightarrow h_d(t) \Leftrightarrow H_d(f) \rightarrow n_o(t) \]

Proof:

\[ n_o(t) = n(t) \ast h_d(t) \]

\[ R_{n_o}(t, t + \tau) = E[n_o(t)n_o(t + \tau)] \]

\[ = E \left[ \int_{-\infty}^{\infty} n(t - \alpha)h_d(\alpha)d\alpha \int_{-\infty}^{\infty} n(t + \tau - \beta)h_d(\beta)d\beta \right] \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_n(\tau - \beta + \alpha)h_d(\beta)d\beta \right] h_d(\alpha)d\alpha \]

\[ = \int_{-\infty}^{\infty} \left[ R_n(\tau + \alpha) \ast h_d(\tau + \alpha) \right] h_d(\alpha)d\alpha \]

\[ = R_n(\tau) \ast h_d(\tau) \ast h_d(-\tau) = R_{n_o}(\tau) \]
White Gaussian Noise

Properties

\[ S_{n_0}(f) = \left| H_d(f) \right|^2 S_n(f) \]

Proof (cont.):

\[ S_{n_0}(f) = \mathcal{F}[R_{n_0}(\tau)] = \mathcal{F}[R_n(\tau) * h_d(\tau) * h_d(-\tau)] \]
\[ = S_n(f) \cdot H_d(f) \cdot H_d^*(f) \]
\[ = S_n(f) \cdot |H_d(f)|^2 \]

\[ P_{n_o} = R_{n_0}(0) = \int_{-\infty}^{\infty} S_n(f) \cdot |H_d(f)|^2 \cdot e^{j2\pi f \cdot 0} \, df \]
\[ = \int_{-\infty}^{\infty} S_n(f) \cdot |H_d(f)|^2 \, df \]
Model of Physical Layer Communications

Model

![Block Diagram of Physical Layer](image)

- **Symbol “1″ →** \( S_1(t) = S(t) \)
- **Symbol “0″ →** \( S_0(t) = -S(t) \)

Where \( S(t) \) is a real time function (waveform) with duration \( T \) \((S(t), t \notin [0, T])\)

ii. **AWGN Channel**

The received signal is \( y(t) = S_i(t) * \delta(t) + n(t) = S_i(t) + n(t) \), \( i = 1, 2 \), where \( n(t) \) is additive noise with psd \( S_n(f) \).
iii. The decision is made based on the output of the detector $r(t)$ at time $t = T$

$$r(t) = y(t) * h_d(t) = [S_i(t) + n(t)] * h_d(t)$$

$$= S_i(t) * h_d(t) + n(t) * h_d(t) = S_{oi}(t) + n_o(t)$$

$r(t)$ consists of both signal and noise information

Taking a sample of $r(t)$ at $t = T$, $r(T) = S_{oi}(T) + n_o(T)$

Note: $r(T)$ is a Gaussian random variable because of $n_o(t)$

Taking a sample at $T$ is to make use of all the signal information over the symbol interval

The design goal of the optimum detector is to find $h_d(t)$, so that the SNR of $r(T)$ is maximized
Signal to Noise Ratio

- Signal to Noise Ratio
  \[
  (SNR)_o = \frac{[S_o(T)]^2}{E[n_o^2(T)]}
  \]

- It has been proved that in order to maximize the signal-to-noise ratio, the design must follow:
  \[
  h_d(t) \Leftrightarrow H_d(f) \\
  h_d(t) = AS(T - t)
  \]

- \(h_d(t)\) is matched to the time revered and delayed version of \(S(t)\).
  This implies that the optimum detector is a **Matched Filter**. This applies to **AWGN** channel models.
Examples

1. \( S(-t) \)

\[ S(t) = \frac{at}{T} \]

\[ h_d(t) = S(T - t) \]

\[ h_d(t) = S(T - t) \]

\[ S_o(t) = S(t) * h_d(t) \]

2.

\[ S_o(t) = S(t) * h_d(t) \]

\[ \frac{a^2T}{3} \]

\[ \frac{a^2T}{3} \]
Signal Energy

- For $S_1(t) = S(t)$ and $S_2(t) = -S(t)$

$$S_O(T)|_{1^n} = \frac{2}{N_O} E_s \quad S_1(t) = S(t)$$

$$S_O(T)|_{0^n} = -\frac{2}{N_O} E_s \quad S_2(t) = -S(t)$$

where

$$S(t) = \begin{cases} a, & t \in [0,T] \\ 0, & t \notin [0,T] \end{cases}$$

- Signal energy $E_s$ is given by:

$$E_s = \int_0^T S^2(t) \, dt = \int_0^T S^2(T-t) \, dt$$

- The noise component is:

$$S_{n_o}(f) = |H_d(f)|^2 S_n(f) = \frac{N_O}{2} |H_d(f)|^2$$

Which is not constant, it is NOT White Gaussian Noise, it is simply Gaussian Noise.
Noise Energy

First, review Parseval’s Theorem:

\[ \int_{-\infty}^{\infty} x^2(t) \, dt = \int_{-\infty}^{\infty} \left| X(f) \right|^2 \, df \]

Then:

\[ \sigma_{n_0}^2 = E \left[ n_0^2(t) \right] = R_{n_0}(0) = \int_{-\infty}^{\infty} S_{n_0}(f) \, df \]

\[ = \int_{-\infty}^{\infty} \frac{N_0}{2} \left| H_d(f) \right|^2 \, df \]

\[ = \int_{0}^{T} \frac{N_0}{2} h^2_d(t) \, dt \quad \text{(From the Parseval’s Theorem)} \]

\[ = \frac{N_0}{2} \int_{0}^{T} \left( \frac{2}{N_0} \right)^2 S^2(T-t) \, dt \]

\[ = \frac{2}{N_0} E_s \]
Average Signal

- **Calculation of Average Signal Value:**

\[
S_O(T) = \frac{S_O(T)_{1^\text{st}} - S_O(T)_{0^\text{th}}}{2} = \frac{2E_S}{N_o}
\]

Therefore:

\[
(SNR)_O = \left(\frac{2E_S}{N_o}\right)^2 = \frac{2E_S}{N_o} = \frac{\text{signal energy}}{\text{noise power spectral density}}
\]

\[
= \frac{E_S/T}{N_o/2/T} = \frac{P_S}{P_{no}} = \frac{\text{signal power}}{\text{noise power}}
\]
Decision Threshold

- Here the sampled values: \( r(T) \) is a Gaussian random variable. The following notation denotes that \( r(T) \) is a Gaussian random variable with a (mean, variance).

\[
r(T)|_{1} \sim N\left(\frac{2E_S}{N_0}, \frac{2E_S}{N_0}\right) \quad r(T)|_{0} \sim N\left(\frac{-2E_S}{N_0}, \frac{2E_S}{N_0}\right)
\]

- Let \( \mu = r(T) \), then:

\[
\mu_1 = \frac{2E_S}{N_0} \quad \mu_0 = \frac{-2E_S}{N_0}
\]

\[
\mu_d = \frac{\mu_1 + \mu_2}{2} = 0
\]

If \( u \geq (\mu_d = 0) \), then bit “1” was sent

If \( u < (\mu_d = 0) \), then bit “0” was sent
Error Events

- An error occurs in the following cases:
  
  \[ U < 0|'1' \text{ and } U \geq 0|'0' \]

  \[ P_{\text{error}} = P(u < 0|'1') P('1') + P(u \geq 0|'0') P('0') \]

  Since \( P('1') = P('0') = \frac{1}{2} \)

  \[ P_{\text{error}} = \frac{1}{2} \left[ P(u < 0|'1') + P(u \geq 0|'0') \right] \]

  \[ = \frac{1}{2} \left[ \phi \left( \frac{0 - \frac{2E_S}{N_o}}{\sqrt{\frac{2E_S}{N_o}}} \right) + 1 - \phi \left( \frac{0 - \left( -\frac{2E_S}{N_o} \right)}{\sqrt{\frac{2E_S}{N_o}}} \right) \right] \]

  \[ = 1 - \phi \left( \sqrt{\frac{2E_S}{N_o}} \right) = Q \left( \sqrt{\frac{2E_s}{N_o}} \right) = Q \left( \sqrt{(SNR_o)} \right) \]
Shannon Channel Capacity

- The maximum reliable transmission rate over an ideal channel with bandwidth $W$ Hz, with Gaussian distributed noise, and with SNR is

$$C = W \log_2(1 + SNR)$$

- Reliable means that the error rate can be made arbitrarily small by using proper coding
Example

Consider a telephone line with the frequencies from 300Hz to 3400Hz and with a SNR of 35dB. What is the Shannon channel capacity?

Since

\[ SNR(db) = 10 \log_{10} SNR \]

\[ SNR = 10^{35/10} \]

Therefore,

\[ C = W \log_2 (1 + SNR) \]

\[ = (3400 - 300) \log_2 (1 + 10^{3.5}) \]

\[ = 36044 \text{ bps} \]
Physical Layer (general signal)

- Model

**Block Diagram of Physical Layer**

Transmitter: bit -> symbol

Channels: AWGN, the simplest channel

Receiver: demodulator/detector, sampling, decision device
Binary signaling: two signals with symbol duration $T$
- “0” $\rightarrow s_0(t)$ with probability $P(b=0)=0.5$
- “1” $\rightarrow s_1(t)$ with probability $P(b=1)=0.5$

The average signal energy is

$$E_s = \frac{1}{2} \int_0^T \left[s_0^2(t) + s_1^2(t)\right] dt$$
AWGN channel model:
- **Channel response**: \( h_c(t) = \delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \)
- **Fourier transform**

\[
H_c(f) = F(h_c(t)) = \int_{-\infty}^{\infty} h_c(t) e^{-2\pi ft} dt = 1
\]

(Channel of infinite bandwidth)

**Received signal**
\[
r(t) = s_i(t) \otimes h_c(t) + n(t) = s_i(t) + n(t)
\]

(convolution)
How to detect what 'bit' has been transmitted from the received signal \( r(t) \)?

What is the probability of detection error?

Objective:

- Design a detection and decision devices to minimize

\[
P_e = P\{\hat{b} \neq b\}
\]
Detector (matched filter)

In order to minimize $P_e$, the impulse response of the detector is matched to the time-reversed version of input signal

$$h_d(t) = s_1(T - t) - s_0(T - t)$$
Output of the detection:

\[ y(t) = r(t) \otimes h_d(t) = (s_i(t) + n(t)) \otimes h_d(t) \]

\[ = \int_{-\infty}^{\infty} s_i(\tau) h_d(t - \tau) d\tau + \int_{-\infty}^{\infty} n(\tau) h_d(t - \tau) d\tau \]

The decision variable

\[ y(T) = \int_{-\infty}^{\infty} s_i(\tau) h_d(T - \tau) d\tau + \int_{-\infty}^{\infty} n(\tau) h_d(T - \tau) d\tau \]

\[ = \int_{-\infty}^{\infty} s_i(\tau)(s_1(\tau) - s_0(\tau)) d\tau + \int_{-\infty}^{\infty} n(\tau)(s_1(\tau) - s_0(\tau)) d\tau \]

Signal component \quad Noise component
Mean of $Y(T)$

$$E[y(T)] = E\left[\int_{-\infty}^{\infty} s_i(\tau)(s_1(\tau) - s_0(\tau))d\tau\right] + E\left[\int_{-\infty}^{\infty} n(\tau)(s_1(\tau) - s_0(\tau))d\tau\right]$$

$$= \int_{-\infty}^{\infty} s_i(\tau)(s_1(\tau) - s_0(\tau))d\tau + \int_{-\infty}^{\infty} E[n(\tau)](s_1(\tau) - s_0(\tau))d\tau$$

$$= \int_{-\infty}^{\infty} s_i(\tau)(s_1(\tau) - s_0(\tau))d\tau$$

$$\mu_i$$
\( \textbf{Variance of } Y(T) \)

\[ V[y(T)] = E\{[y(T) - E(y(T))]^2 \} \]

\[ = E\{[\int_{-\infty}^{\infty} n(\tau)(s_1(\tau) - s_0(\tau))]^2 d\tau \} \]

\( \delta^2 \)

\( y(T) \sim N(\mu_i, \delta^2) \)  
Normal distribution the mean \( \mu_i \) and variance \( \delta^2 \)
Hypothesis testing

- **Given bit “0” was sent**
  \[ f(y(T) \mid b = 0) = N(\mu_0, \delta^2) \]

- **Given bit “1” was sent**
  \[ f(y(T) \mid b = 1) = N(\mu_1, \delta^2) \]
**Decision Rule**

- **Assuming** $\mu_0 < \mu_1$ \quad $z = y(T)$

\[ f(z | b = 0) \quad f(z | b = 1) \]

- **Threshold** $\alpha = \frac{\mu_0 + \mu_1}{2}$
  - If $z = y(T) \geq \alpha$, then $\hat{b} = 1$, i.e., $b = 1$ is regarded to be sent
  - If $z = y(T) < \alpha$, then $\hat{b} = 0$, i.e., $b = 0$ is regarded to be sent
Bit Error (1/3)

Error event: an error occur in the following cases:

- When $z < \alpha$ (i.e., $\hat{b} = 0$), but $b = 1$ was sent
- When $z \geq \alpha$ (i.e., $\hat{b} = 1$), but $b = 0$ was sent
Bit Error (2/3)

**Bit Error Probability**

\[ P_e = P(\hat{b} = 0 \mid b = 1)P(b = 1) + P(\hat{b} = 1 \mid b = 0)P(b = 0) \]

\[ = P(z < \alpha \mid b = 1)P(b = 1) + P(z \geq \alpha \mid b = 0)P(b = 0) \]

Assuming that \( P(b = 1) = P(b = 0) = \frac{1}{2} \)

\[ P_e = \frac{1}{2} \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\delta^2} e^{-\frac{(x-\mu_1)^2}{2\delta^2}} \, dx + \frac{1}{2} \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}\delta^2} e^{-\frac{(x-\mu_0)^2}{2\delta^2}} \, dx \]

\[ = \frac{1}{2} \Phi\left( \frac{\alpha - \mu_1}{\delta} \right) + \frac{1}{2} \left[ 1 - \Phi\left( \frac{\alpha - \mu_0}{\delta} \right) \right] \]

where \( \Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \) and \( \Phi(-y) = 1 - \Phi(y) \) due to symmetry.
Bit Error Probability

\[ P_e = \frac{1}{2} \Phi \left( \frac{\alpha - \mu_1}{\delta} \right) + \frac{1}{2} \left[ 1 - \Phi \left( \frac{\alpha - \mu_0}{\delta} \right) \right] \]

Since \( \alpha = \frac{\mu_0 + \mu_1}{2} \)

\[ P_e = \frac{1}{2} \Phi \left( \frac{\mu_0 - \mu_1}{2\delta} \right) + \frac{1}{2} \left[ 1 - \Phi \left( \frac{\mu_1 - \mu_0}{2\delta} \right) \right] = 1 - \Phi \left( \frac{\mu_1 - \mu_0}{2\delta} \right) = Q \left( \frac{\mu_1 - \mu_0}{2\delta} \right) \]

\((Q(x) = 1 - \Phi(x))\)
Signal to Noise Ratio (SNR)

\[ SNR = \frac{\text{signal power}}{\text{noise power}} = \frac{[s(T)]^2}{P_{no}} \]

where \( s(T) \): average amplitude of the received signal

\[ y(T) = \int_{-\infty}^{\infty} s_i(\tau)(s_1(\tau) - s_0(\tau))d\tau + \int_{-\infty}^{\infty} n(\tau)(s_1(\tau) - s_0(\tau))d\tau \]
Signal Power

\[ s(T) = |E(y(T))|_{b=0} \times P(b = 0) + |E(y(T))|_{b=1} \times P(b = 1) \]

\[ = \frac{1}{2} |\mu_0| + \frac{1}{2} |\mu_1| = \frac{1}{2} (\mu_1 - \mu_0) \quad \text{(assuming that } \mu_1 > 0 \geq \mu_0) \]

\[ = \frac{1}{2} \int_0^T \left[ s_1^2(\tau) - 2s_1(\tau)s_0(\tau) + s_0^2(\tau) \right] d\tau \quad (\mu_i = \int_{-\infty}^{\infty} s_i(\tau)(s_1(\tau) - s_0(\tau))d\tau) \]

\[ = E_s - E_s \rho = (1 - \rho)E_s \quad (\frac{1}{2} (\mu_1 - \mu_0) = (1 - \rho)E_s) \]

where \[ E_s = \frac{1}{2} \int_0^T [s_1^2(\tau) + s_0^2(\tau)] d\tau \]

\[ \rho = \frac{1}{E_s} \int_0^T [s_1(\tau)s_0(\tau)] d\tau \]
**Noise Power**

$$P_{n0} = E \left[ \left( \int_{-\infty}^{\infty} n(\tau) \left( s_1(\tau) - s_0(\tau) \right) d\tau \right)^2 \right] = \delta^2$$

$$= E \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(\tau)(s_1(\tau) - s_0(\tau))n(\omega)(s_1(\omega) - s_0(\omega))d\tau d\omega \right]$$

$$= \frac{N_0}{2} \int_{0}^{T} (s_1(\tau) - s_0(\tau))^2 d\tau$$  \hspace{1cm} (due to \( R_n(\tau) = \frac{N_0}{2} \delta(\tau) \))

$$= N_0 E_s - N_0 \rho E_s = (1 - \rho) N_0 E_s$$
Performance metrics

SNR:

$$SNR = \frac{[s(T)]^2}{P_{n0}} = \frac{1-\rho}{N_0} E_s$$

Probability of error:

$$P_e = Q\left(\frac{\mu_1 - \mu_0}{2\delta}\right) = Q(\sqrt{SNR}) = Q\sqrt{\frac{1-\rho}{N_0} E_s}$$

$$\left(\frac{1}{2}(\mu_1 - \mu_0) = (1 - \rho)E_s \right)$$

$$\left(\delta^2 = N_0E_s - N_0\rho E_s = (1 - \rho)N_0E_s \right)$$
What is Line Coding?

- Mapping of binary information sequence into the digital signal that enters the channel
  - Ex. “1” maps to +A square pulse; “0” to −A pulse
- Baseband modulated signal has form

\[ x(t) = \sum_k a_k p(t - kT_b) \]

where \( p(t) \) is the pulse of duration \( T_b \) (the bit time) with amplitude \( a_k \)
  - e.g, \( a_k = A \) for a “1” bit and \( 0 \) for a “0” bit
  - The data rate is \( 1/T_b \)

<table>
<thead>
<tr>
<th>Bit Sequence</th>
<th>Unipolar NRZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 1 0 1 1 1 1 0 0</td>
<td></td>
</tr>
</tbody>
</table>
Line coding examples

Unipolar NRZ

Polar NRZ

NRZ-inverted (differential encoding)

Bipolar encoding

Manchester encoding

Differential Manchester encoding
The Band-Limited Model

• In the case of the band-limited model the channel does not have infinite bandwidth and thus, the time-domain is not an impulse at $t=0$. As such, the convolution causes an Inter-Symbol Interference (ISI).

• For example, if the impulse response is:

![Impulse response diagram]

Send “1”

Send “0”

0 T 2T

0 T 2T

• And the following data is transmitted:

<table>
<thead>
<tr>
<th>“1”</th>
<th>“0”</th>
<th>“1”</th>
<th>“1”</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2T</td>
<td>3T</td>
<td>4T</td>
</tr>
</tbody>
</table>
The Band-Limited Model

• The result is the linear superposition of the impulse responses. The individual responses and the net summation are shown below.

• In order to accomplish this, the Modular is split into two parts: the pre-coder and the pulse-shaping filter. The pulse shaping filter generates a unit pulse only. The inputs to the pre-coder are the “1” and “0” bits. The output is “d” and “-d”.

\[ a_n \in \{"0","1"\} \quad b_n \in \{"-d","d"\} \]

The pre-coder transforms \{a_n\} to a desired form \{b_n\} for baseband transmission.
## Pulse Shapes

<table>
<thead>
<tr>
<th>Polar (anti-Polar) Shaping</th>
<th>Uni-polar Shaping</th>
<th>Bi-polar Shaping</th>
<th>Manchester Shaping</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_n = \begin{cases} d, &amp; &quot;1&quot; \ -d, &amp; &quot;0&quot; \end{cases}$</td>
<td>$b_n = \begin{cases} d, &amp; &quot;1&quot; \ 0, &amp; &quot;0&quot; \end{cases}$</td>
<td>$b_n = \begin{cases} +d / -d, &amp; &quot;1&quot; \ 0, &amp; &quot;0&quot; \end{cases}$</td>
<td>$b_n = \begin{cases} +d &amp; &quot;1&quot; \ -d &amp; &quot;0&quot; \end{cases}$</td>
</tr>
</tbody>
</table>

\[
X(t) = \sum b_n h_T(t - nT)
\]
\[
y(t) = X(t) * h_c(t) = \sum b_n g(t - nT)
\]
\[
g(t - nT) = h_T(t - nT) * h_c(t)
\]

### Observations:

1. The duration of the transmitted pulse is ‘stretched’ through a channel with memory.
2. The pulse stretching is referred to as “time dispersion”, and the channel is called a “time dispersion channel.”
3. Time dispersion causes overlaps between adjacent symbols at the output of the channel. The overlaps are called **Inter-Symbol Interference (ISI).**
4. ISI can’t be suppressed by simply increasing the signal energy.
The above figure represents the band-limited model for the physical layer. From this, the following equations are obtained:

\[ x(t) = \sum_n b_n h_T(t - nT) = \sum_n b_n [\delta(t - nT) * h_T(t)] \]

The output of the convolution is:

\[ r(t) = \left[ \sum_n b_n \delta(t - nT) \right] * h_T(t) * h_c(t) * h_d(t) \]

Let \( f(t) = h_T(t) * h_c(t) * h_d(t) \), then:

\[ \sum_n b_n f(t - nT) = \sum_n b_n \mu c(t - nT) \]

where \( \mu \) is a scaling factor such that \( c(0) = 1 \).
ISI Free Samples

\[ \mu c(t) = h_T(t) \ast h_c(t) \ast h_d(t) \]

\[ \mu C(f) = H_T(f)H_c(f)H_d(f) \]

with \[ \int_{-\infty}^{\infty} C(f) df = \int_{-\infty}^{\infty} C(f)e^{j2\pi f \cdot 0} df = c(0) = 1 \]

• \( \mu C(f) \) or \( \mu c(t) \) is the transfer function of the effective channel.

\[ r(t) = \mu \sum_n b_n c(t - nT) \]

• The sampled \( r(t) \) value at \( t_i = iT \) is:

\[ r(t_i) = \mu \sum_n b_n c(iT - nT) = \mu b_i c(0) + \mu \sum_{n \neq i} b_n c(iT - nT) \]

\[ = \mu b_i + ISI \]
To achieve ISI free samples:

\[ c(iT - nT) = 0, \quad n \neq i \]

\[ c(t_i) = 0, \quad t_i = ..., -2T, -T, T, 2T, ... \]

or equivalently:

\[ c(iT - nT) = \begin{cases} 1, & n = i \\ 0, & n \neq i \end{cases} \]
Frequency Domain Condition

- Consider sampling $c(t)$ at $t_k = kT$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

\[
c_\delta(t) = c(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} c(kT) \delta(t - kT)
\]

\[
c_\delta(f) = R \sum_{n=-\infty}^{\infty} C(f - nR)
\]

where $R = 1/T$ is the bit rate.

\[
C_\delta(f) = \mathcal{F}\left[c_\delta(t)\right] = \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c(kT) \delta(t - kT)\right] = \sum_{k=-\infty}^{\infty} c(kT) \mathcal{F}\left[\delta(t - kT)\right] = \sum_{k=-\infty}^{\infty} c(kT) e^{-j2\pi kT}
\]

- For ISI free samples:

\[
c(iT - nT) = 1 \implies i = n
\]

\[
k = i - n \implies k = 0
\]

\[
R \sum_{n=-\infty}^{\infty} (f - nR) = 1
\]
Ideal Nyquist Channel

• Ideal Nyquist Channel

\[ C(f) = \begin{cases} 
\frac{1}{2B_0}, & |f| \leq B_0 \\
0, & |f| > B_0 
\end{cases} \]

where \( R = 2B_0 \)

• \( B_0 \) is the effective channel single-side bandwidth. Here the ISI Free Sample Condition gives the following:

\[ \sum_{m=-\infty}^{\infty} C(f - mR) = T \]

| m | -2 | -1 | 0 | 1 | 2 | ...
---|----|----|---|---|---|---
\[ \frac{1/(2B_0)}{1/(2B_0)} \]

• Practical Limitations

1. an ideal Low-pass filter is not physically realizable

2. \( c(t) = \mathbb{F}^{-1}[C(f)] = \frac{\sin(2\pi B_0 t)}{2\pi B_0 t} = \text{sinc}(2B_0 t) \propto \frac{1}{|t|} \)

\( c(t) \) decreases slowly as \(|t|\) increases
ISI Free Samples

\[ p(t) = \text{sinc}^2(10^6 t) \]

Figure 7.9  A series of sinc pulses corresponding to the sequence 1011010.
Raised Cosine Spectrum

- To overcome the limitation, a particular pulse spectrum that has desirable spectral properties and has been readily used in practice is the raised cosine spectrum.

\[
C(f) = \begin{cases} 
\frac{1}{2B_0}, & 0 \leq |f| \leq [(1 - \alpha)B_0] \\
\frac{1}{4B_0} \left[1 + \cos \left(\frac{\pi}{2\alpha B_0}(|f|-(1-\alpha)B_0)\right)\right], & (1-\alpha)B_0 \leq |f| \leq (1+\alpha)B_0 \\
0, & |f| \geq (1+\alpha)B_0
\end{cases}
\]

Here, taking the inverse Fourier Transform gives:

\[
c(t) = \mathcal{F}^{-1}\left[ C(f) \right] = \text{sinc}(2B_0t) \left[ \cos(2\pi\alpha B_0 t) \right] \left[ \frac{1}{1-16\alpha^2 B_0^2 t^2} \right] \propto \frac{1}{t^3}
\]

\(\alpha \in [0,1]\) is called the roll-off factor. \(w = (1 + \alpha)B_0\) is the single-sided bandwidth.

- **Advantages:**
  - \(C(f)\) is practically realizable
  - \(c(t)\) decreases proportionally to \(t^{-3}\)

- **Disadvantages:**
  - Lower efficiency of frequency spectrum.
Raised Cosine Spectrum

Figure 6.7 Responses for different rolloff factors. (a) Frequency response. (b) Time response. Note that $B_o = 1/2T_b$. 

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Summary of Chapter 3

Given the characteristics of the communication channel (AWGN, Bandlimited) and system resources (signal power, bandwidth), we can calculate the probability of transmission error (bit error rate), i.e., the required performance.