

Instructions:

1. The exam will be graded out of 58.
2. If there is insufficient room, there is a blank page at the end of the examination. Clearly indicate your solution continues on that page.
3. Turn off all electronic media and store them under your desk.
4. You may ask only one question during the examination: "May I go to the washroom?"
5. Asking any other question will result in a deduction of 5 marks from the exam grade.
6. If you think a question is ambiguous, write down your assumptions and continue.
7. Do not leave during first hour or after there are only 15 minutes left.
8. Do not stand up until all exams have been picked up.
9. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.

1. (1 point) The roots of the polynomial $x^3 + (-1 + 2j)x^2 + (-5 - 3j)x + (6 - 2j)$ are $1 - j$, $-2 - j$ and 2. What are the roots of the polynomial $(1 + j)x^3 + (-3 + j)x^2 + (-2 - 8j)x + (8 + 4j)$?

2. (2 points) Given a real number x , is it always true that $|2x| = |x + xj|$? Justify your answer. If it is always true, you must prove it is true for all x , but if it is false, you only need to show one counter example.

3. (4 points) Perform Gaussian elimination with partial pivoting to find the equivalent matrix in row-echelon form on the following matrix:

$$\begin{pmatrix} -0.3 & -0.2 & 3.9 & 5.2 \\ -1.5 & -1 & 0.3 & 2 \\ 3 & 2 & 1 & -2 \end{pmatrix}.$$

4. (3 points) For each of the following matrices, find the inverse:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} -0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}.$$

5. (2 points) Demonstrate that if a real matrix is such that $A^\top = A^{-1}$, then $\|A\mathbf{u}\|_2 = \|\mathbf{u}\|_2$ for all vectors \mathbf{u} . You should recall that $\|\mathbf{u}\|_2^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.

6. (3 points) Suppose you want to find that linear combination of the two vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \end{pmatrix}$ that best approximates the vector $\begin{pmatrix} 0.3 \\ 0.4 \\ 0.7 \\ 0.6 \end{pmatrix}$. What is the augmented matrix that describes the system of linear equations to find the coefficients of this best approximation? You do not have to solve this system of linear equations.

7. (3 points) Recalling that for real matrices A and B , $(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$, that $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$, and that real matrices of the same dimension form a vector space, show that for a square real matrix A that $A + A^\top$ is self-adjoint (that is, symmetric).

8. (5 points) Suppose you determined that the real matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 3 & 2 \\ -1.14 & -0.57 & 5 & 0.29 & -2.14 \\ 0.6 & 0.3 & 3.05 & 2.12 & 1.99 \\ -0.34 & -0.17 & 0.5 & -0.31 & 0.56 \end{pmatrix}$$

is equivalent to the matrix in row-echelon form

$$\begin{pmatrix} 2 & 1 & 0 & 3 & 2 \\ 0 & 0 & 5 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

What are the domain and co-domain of A ? What is the dimension of the range of A and what is a basis for that range? What is a basis for the null space of A ?

9. (3 points) Recall that a linear operator A is self-adjoint if $A^* = A$, it is isometric if $A^* = A^{-1}$, and it is skew-adjoint if $A^* = -A$. What are the analogous properties for complex numbers and their conjugates, and describe complex numbers satisfy each of these analogous properties?

10. (1 point) Write down the adjoint of this matrix:

$$A = \begin{pmatrix} 1+j & 2-j \\ -3 & 4j \\ 5+6j & -7-8j \end{pmatrix}$$

11. (1 point) True or false: A is self-adjoint if and only if jA is skew-adjoint.

12. (4 points) You know that

$$A = \begin{pmatrix} 0.3 & 0.9 & 2.6 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0.3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Use this decomposition to find all solutions $A\mathbf{u} = \begin{pmatrix} 3.8 \\ 6 \end{pmatrix}$.

13. (2 points) Show that if \mathbf{u}_1 is a solution to $A\mathbf{u} = \mathbf{b}$ and that if \mathbf{v}_1 and \mathbf{v}_2 are vectors in the null space of A , then $\mathbf{u}_1 + \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ is also a solution to $A\mathbf{u} = \mathbf{b}$.

14. (5 points) For a matrix $A : \mathbf{R}^n \rightarrow \mathbf{R}^m$, which of the following statements are equivalent (describe the same situation):

1. A is onto.
2. $m < n$.
3. A cannot be one-to-one.
4. $m > n$.
5. A is either one-to-one and onto, or neither.
6. The null space is just the zero vector $\mathbf{0}_n$.
7. $m = n$.
8. The rank of A is equal to m .
9. A cannot be onto.
10. A is one-to-one.

Indicate equivalence by writing those numbers that are equivalent in vectors. For example, if you think 1, 3, and 5 are equivalent, then write $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

15. (3 points) This matrix has eigenvalues 1, 2, 3 and 4:

$$A = \begin{pmatrix} 3.5 & -1.5 & -0.5 & -0.5 \\ -0.5 & 2.5 & -0.5 & -0.5 \\ 0.5 & -1.5 & 2.5 & -0.5 \\ 0.5 & -1.5 & 0.5 & 1.5 \end{pmatrix}.$$

How many eigenvectors are associated with the eigenvalue 3 and what is a system of linear equations that must be solved to find that eigenvector or eigenvectors? How many free variables will be associated with the system of linear equations you wrote down?

16. (3 points) The matrix

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

has eigenvalues 2, 1 and 3. Find an eigenvector corresponding to the eigenvalue 3.

17. (3 points) You have found that the following two are eigenvectors corresponding to a given eigenvalue: $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$. Find an orthonormal basis for the eigenspace corresponding to that eigenvalue. You may wish to note that $56^2 + 42^2 = 4900$ and that multiples of 7 are 7, 14, 21, 28, 35, 42, 49, 56, 63, 70.

18. (2 points) If real vectors \mathbf{u} and \mathbf{v} are orthogonal and if a real matrix A is isometric (orthogonal), show that $A\mathbf{u}$ is orthogonal to $A\mathbf{v}$ using the properties of the inner product and the fact that $A^* = A^{-1}$.

19. (3 points) If an $n \times n$ square matrix A is real, then $\det(\lambda I - A)$ must be a polynomial of degree n in λ with real coefficients. How many eigenvalues (counting multiplicity) does A have, and what are some of the restrictions on those eigenvalues?

20. (2 points) What are the determinants of the follow two matrices:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}?$$

21. (3 points) Define the span of a collection of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$, and describe how you would find a basis for the span of those vectors. Given the n vectors, if we define the matrix $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, is the span of the n vectors equal to the range of A ?