Instructions:

1. At an time, you may use any of the axioms of fields, vector spaces, norms or inner products to simplify your calculations to get to a solution.
2. If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
3. The exam will be graded out of 38, but read the entire examination first.
4. If you require extra room, you can use the last page, but indicate at the question that your answer continues on that page.
5. Turn off all electronic media and store them under your desk or at the front of the examination room.
6. You may ask only one question during the examination: "May I go to the washroom?"
7. Asking **any** other question **will** result in a deduction of five (5) marks from the examination grade.
8. Do not stand up until all examinations have been collected and you have been instructed to leave.
9. Do not leave during first hour or after there are only 15 minutes left.
10. You may rip off the formula page, the last page, as soon as you read this.

1. (3 points) [Alex] We can prove that the multiplicative inverse in a field is unique by assuming that one multiplicative inverse α^{-1} exists, but if $\alpha x = 1$, then multiplying both sides by α^{-1}, we get $(\alpha x)\alpha^{-1} = 1 \cdot \alpha^{-1}$. Using commutativity and associativity on the left hand side and the property of 1 on the right, we deduce that $x = \alpha^{-1}$, so the multiplicative inverse must be unique.

   Prove that $(\alpha\beta)^{-1} = \alpha^{-1}\beta^{-1}$. That is, to find the multiplicative inverse of the product $\alpha\beta$, you need only multiple the multiplicative inverses of α and β. You can show this is the inverse by multiply $\alpha\beta$ by the right-hand side, and then showing each step (using one axiom per step) to show that this product ultimately equals 1. Consequently, using the previous result, because $(\alpha\beta)(\alpha^{-1}\beta^{-1}) = 1$, the right hand side must be $(\alpha\beta)^{-1}$.

2. (1 point) As $v = iz$ where z is the complex impedance, if $z = 3 + 4j$, what will $|v|$ equal if $|i| = 7$?

3. (2 points) Find the rectangular representation $\alpha + \beta j$ of the ratio $\frac{3+j}{1+7j}$.

4. (2 points) If $z = 0.12 - 0.16j$ has the property that $z^{-1} = 3 + 4j$, find the rectangular representation $\alpha + \beta j$ of $\left(z^2\right)^{-1}$.

5. (1 point) Given a real number θ, is it always true that $|\cos(\theta) + \sin(\theta)| = |\cos(\theta) + \sin(\theta)j|$? Justify your answer. If it is always true, you must prove it is true for all θ, but if it is false, you need show only one counter example.

6. (1 point) Simplify the ratio $\frac{w(3-4j+(1+j)z)z^*}{(z(3+4j+(1-j)z^*)w^*)^*}$ as much as possible.

7. (3 points) How many roots (counting multiplicity) does the polynomial x^4+5x^2+4 have? How many of those roots are real, and justify your answer with an argument. If there are complex roots, how are they related to each other?

8. (2 points) If $r = 0.88 - 0.16j$, what is $\sum_{k=0}^{\infty} r^k$ in the the rectangular representation $\alpha + \beta j$? Note that $|r|^2 = 0.8$. Hint: see Question 4, but use that result carefully.

9. (3 points) [Alex] The collection of all complex-valued functions of a real variable is a vector space.

   A complex-valued function of a real variable $f(x)$ is said to be "even" if $f(-x) = f(x)$ for all real x. For example, x^2, $\cos(x)$, e^{-x^2}, and $jx^4 + (-2+4j)x^2 + 7 - 5j$ are all even functions. Is the collection of all even functions a subspace of all complex-valued functions of a real variable?

   For this question, if it is a subspace, you need to provide an argument for why it is, but not necessarily a "proof." If it is not a subspace, you only need to find one counter-example that shows that it is not a subspace.

10. (3 points) [Alex] Using the axioms of a norm, show that $\|3.2\mathbf{u} - 4.7\mathbf{v}\| \leq 3.2\|\mathbf{u}\| + 4.7\|\mathbf{v}\|$. Be sure to apply only one axiom per step.

    Next, show that equality may hold by calculating both the left-hand and right-hand sides of this inequality when $\mathbf{u} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$. You will note that $\mathbf{v} = -2\mathbf{u}$.

11. (3 points) Argue that the infinity-norm of an n-dimensional vector must always be less-than or equal to the one-norm. You don't have to prove, you simply need to provide a reasonable argument, possibly with an example.

    Find one vector $\mathbf{u} \in \mathbf{R}^3$ where $\|u\|_1 = \|u\|_\infty$.

    Recall that $\|\mathbf{u}\|_1 = |u_1| + \cdots + |u_n|$ while $\|\mathbf{u}\|_\infty = \max\{|u_1|, \ldots, |u_n|\}$.

12. (4 points) For two real vectors in $\mathbf{R}^n$, what must be true for $\|\mathrm{proj}_{\mathbf{u}}(\mathbf{v})\|_2 = \|\mathrm{proj}_{\mathbf{v}}(\mathbf{u})\|_2$? Recall that:

    1. The inner product is always a real number.
    2. The inner product is symmetric, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

3. $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\langle \mathbf{u},\mathbf{u}\rangle}\mathbf{u}$
4. $\|\mathbf{u}\|_2^2 = \langle \mathbf{u}, \mathbf{u}\rangle$.
5. $\|\alpha\mathbf{u}\|_2 = |\alpha|\|\mathbf{u}\|_2$.
6. If $a \geq 0$ and $b \geq 0$, then $a = b$ if and only $a^2 = b^2$.

13. (2 points) Find one vector that is orthogonal to the vector $\begin{pmatrix} j \\ 1 \end{pmatrix}$. Recall that the definition of the inner product is subtly different for complex vectors than it is for real vectors with the entries of the first vector being conjugated.

14. (3 points) From these two vectors, create a pair of unit vectors that are orthogonal using Gram Schmidt. $\mathbf{u}_1 = \begin{pmatrix} 4 \\ 2 \\ 8 \\ 4 \end{pmatrix}$. $\mathbf{u}_2 = \begin{pmatrix} 4 \\ 12 \\ 14 \\ 12 \end{pmatrix}$.

15. (1 point) Consider the statement: if $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors, then $|\langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle| \leq 1$. Either justify this with a theorem from class, or provide a counter-example in $\mathbf{R}^2$ where this does not hold.

16. (2 points) [Alex] Can two vectors in $\mathbf{R}^3$ that have all positive (greater than zero) entries be orthogonal? Why or why not?

17. (2 points) What is the angle between the two vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$? Recall that the cosine of $0°$, $30°$, $45°$, $60°$ and $90°$ are $\frac{\sqrt{4}}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{1}}{2}$ and $\frac{\sqrt{0}}{2}$, respectively.

This page may be used to continue answering questions, but please indicate that you are continuing the answer on this page.

You may rip off this page.
Multiplication is usually shown by juxtaposition, so $\alpha\beta$ or $\alpha\mathbf{u}$, but when it is necessary to avoid ambiguity, a dot may be used: $1 \cdot \alpha$ or $0 \cdot \mathbf{u}$.

Axioms of a field

Given a collection of scalars $\mathbf{F}$ and two binary operations, scalar addition and scalar multiplication, the following axioms define this collection as a field:

1. A field is closed under addition and multiplication.
2. Addition and multiplication are commutative, meaning $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all α, β in the field.
3. Addition and multiplication are associative, meaning $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ for all α, β, γ in the field.
4. There is an additive identity 0 such that $\alpha + 0 = \alpha$ for all α in the field.
5. There is a multiplicative identity 1 such that $1 \cdot \alpha = \alpha$ for all α in the field.
6. For each α in the field, there is an additive inverse $-\alpha$ such that $\alpha + (-\alpha) = 0$.
7. For each non-zero α in the field, there is a multiplicative inverse α^{-1} such that $\alpha \cdot \alpha^{-1} = 1$.
8. Multiplication distributes over addition, so $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for all α, β, γ in the field.

Axioms of a vector space

Given a collection of vectors $\mathcal{V}$, an associate field $\mathbf{F}$ of scalars and two binary operations, vector addition and scalar multiplication, the following axioms define this collection as a vector space:

1. If $\mathbf{u}$ is a vector, then $\alpha\mathbf{u}$ is also a vector for all $\alpha \in \mathbf{F}$. That is, the vector space is closed under scalar multiplication.
2. $1 \cdot \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{V}$.
3. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ for all $\mathbf{u} \in \mathcal{V}$ and for all $\alpha, \beta \in \mathbf{F}$.
4. If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $\mathbf{u} + \mathbf{v}$ is also a vector. That is, the vector space is closed under vector addition.
5. Vector addition is commutative, meaning $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.
6. Vector addition is associative, meaning $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
7. There exists a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{v}$ for all vectors $\mathbf{v}$.
8. For each $\mathbf{u}$ in the vector space, there is an additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
9. Scalar multiplication distributes over vector addition, so $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ for all scalars α in the field and all vectors $\mathbf{u}, \mathbf{v}$ in the vector space.
10. Scalar multiplication distributes over scalar addition, so $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ for all scalars α, β in the field and all vectors $\mathbf{u}$ in the vector space.

Axioms of an inner product with real scalars

1. It is symmetric, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
2. It is bilinear, meaning it is linear in both entries:
   (a) $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$.
   (b) $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$.
3. It is positive definite, meaning that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Axioms of an inner product with complex scalars

1. It is conjugate symmetric, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$.
2. It is sequilinear (one-and-a-half times linear), meaning it is conjugate linear in the first entry and linear in the second entry:
   (a) $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha^* \langle \mathbf{u}, \mathbf{w} \rangle + \beta^* \langle \mathbf{v}, \mathbf{w} \rangle$.
   (b) $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$.
3. It is positive definite, meaning that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Axioms of a norm

1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$ for all scalars α in the field and all vectors $\mathbf{u}$.
3. It obeys the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for all vectors $\mathbf{u}, \mathbf{v}$.

Other useful properties:

1. $\|u\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$
2. $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$
3. $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$
4. $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ if and only if $|r| < 1$