UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
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NE 112 *Linear algebra for nanotechnology engineering*

1.3 Sequences, sums and series

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Introduction

- In this topic, we will
  - Describe finite sequences and sums
  - Investigate arithmetic sequences and geometric sequences
  - Determine values for arithmetic sums and geometric sums
  - Describe infinite sequences and series
  - Determine values for arithmetic series and geometric series

Sequences and sums

- A finite sequence is a fixed number of values:
  - Suppose we have n temperature readings

$$t_1, t_2, t_3, \ldots, t_n$$

- The sum of this finite sequence is

$$t_1 + t_2 + t_3 + \cdots + t_n$$

- A sum such as this may be written as

$$\sum_{k=1}^{n} t_k$$

Finite sums

- This Greek letter capital-S (*sigma*) is used for *summation notation*

upper limit

$$\sum_{k=1}^{n} t_k$$

term description

index of summation

lower limit

Finite sums

- Just substitute the index of summation with each integer from the lower limit to the upper limit
  - Our lower limit will generally be 0 or 1

$$\sum_{k=1}^{5} t_k = t_1 + t_2 + t_3 + t_4 + t_5$$

  - The term description may be a formula:

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5$$

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$\sum_{k=0}^{5} 8 \cdot 2^k = 8 \cdot 2^0 + 8 \cdot 2^1 + 8 \cdot 2^2 + 8 \cdot 2^3 + 8 \cdot 2^4 + 8 \cdot 2^5$$

Finite sums

- Often, the upper limit will simply be *n*:

$$\sum_{k=1}^{n} t_k = t_1 + t_2 + t_3 + \cdots + t_{n-1} + t_n$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2$$

$$\sum_{k=0}^{n} 8 \cdot 2^k = 8 \cdot 2^0 + 8 \cdot 2^1 + 8 \cdot 2^2 + 8 \cdot 2^3 + \cdots + 8 \cdot 2^{n-1} + 8 \cdot 2^n$$

Arithmetic sequences and series

- A sequence of the form:

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots,\ a + nd$$

  is said to be an arithmetic sequence
  - Each subsequent term is d larger than the previous

- For example,

$$3, 5, 7, 9, 11, 13, 15, \ldots, 101$$

  - In this case, $a = 3$, $d = 2$ and $n = 49$
  - To find n, just solve $3 + 2n = 101$

Arithmetic sequences and series

- An arithmetic sum is the sum of an arithmetic sequence:

$$a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + nd)$$

  - Note that we can factor out $(n + 1)\,a$

$$a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + nd)$$
$$= (n + 1)\,a + (0 + d + 2d + 3d + \cdots + nd)$$

  - Now, we can factor out a d:

$$= (n + 1)\,a + d\,(0 + 1 + 2 + 3 + \cdots + n)$$

$$= (n+1)a + d\sum_{k=0}^{n} k$$

Arithmetic sequences and sums

- Question: what is the value of:

$$\sum_{k=0}^{n} k = 0+1+2+3+\cdots+n \quad ?$$

  - Fortunately, Gauss, discovered this formula when he was seven:

$$\begin{array}{rccccccccc} & 0 + & 1 + & 2 + & 3 & + \cdots + & (n-3) + & (n-2) + & (n-1) + & n \\ + & n + & (n-1) + & (n-2) + & (n-3) & + \cdots + & 3 + & 2 + & 1 + & 0 \\ \hline & n + & n + & n + & n & + \cdots + & n + & n + & n + & n \end{array}$$

$$= n\,(n+1)$$

  - Therefore,

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Arithmetic sequences and sums

- Therefore, the arithmetic sum

$$a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + nd)$$

equals

$$\sum_{k=0}^{n} (a + kd) = a(n+1) + d\,\frac{n(n+1)}{2}$$

$$= (n+1)\left(a + d\,\frac{n}{2}\right)$$

Properties of sums

- We can also deduce some properties of sums:

$$\sum_{k=0}^{n} ba_k = ba_0 + ba_1 + ba_2 + ba_3 + \cdots + ba_n$$
$$= b\left(a_0 + a_1 + a_2 + a_3 + \cdots + a_n\right)$$
$$= b\sum_{k=0}^{n} a_k$$

- Thus, $\sum_{k=0}^{n} ba_k = b\sum_{k=0}^{n} a_k$

Properties of sums

- Also,

$$\sum_{k=0}^{n} 1 = 1 + 1 + 1 + 1 + \cdots + 1$$
$$= (n+1)$$

- Of course, $\sum_{k=1}^{n} 1 = n$

Properties of sums

- Similarly,

$$
\begin{aligned}
\sum_{k=0}^{n}(a_k + b_k) &= (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\
&= (a_0 + a_1 + a_2 + a_3 + \cdots + a_n) + (b_0 + b_1 + b_2 + b_3 + \cdots + b_n) \\
&= \left(\sum_{k=0}^{n} a_k\right) + \left(\sum_{k=0}^{n} b_k\right)
\end{aligned}
$$

Properties of sums

- Can we use these rules to find that

$$\sum_{k=0}^{n}(a+kd)=(n+1)a+d\,\frac{n(n+1)}{2}\ ?$$

- Well,

$$\sum_{k=0}^{n}(a+kd)=\left(\sum_{k=0}^{n}a\right)+\left(\sum_{k=0}^{n}kd\right)$$

$$=a\left(\sum_{k=0}^{n}1\right)+d\left(\sum_{k=0}^{n}k\right)$$

$$=a(n+1)+d\,\frac{n(n+1)}{2}$$

Geometric sequences and sums

- A finite geometric sequence is where each term is the previous term multiplied by a constant ratio of *r*:

$$a, ar, ar^2, ar^3, \ldots, ar^n \qquad ?$$

- A geometric sum is the sum of such a sequence:

$$a + ar + ar^2 + ar^3 + \cdots + ar^n$$

$$= \sum_{k=0}^{n} \left(ar^k \right)$$

Geometric sums

- First, we immediately see that:

$$\sum_{k=0}^{n}\left(ar^k\right) = a\sum_{k=0}^{n} r^k$$

  - Thus, we must ask, what is $\sum_{k=0}^{n} r^k$?

Geometric sums

- To determine this, observe that

$$\sum_{k=0}^{n} r^k = 1 \cdot \sum_{k=0}^{n} r^k$$

$$= \frac{1-r}{1-r} \sum_{k=0}^{n} r^k$$

$$= \frac{1}{1-r} \sum_{k=0}^{n} (1-r) r^k$$

$$= \frac{1}{1-r} \sum_{k=0}^{n} \left(r^k - r \cdot r^k \right)$$

$$= \frac{1}{1-r} \sum_{k=0}^{n} \left(r^k - r^{k+1} \right)$$

$$= \frac{1}{1-r} \left(\left(\sum_{k=0}^{n} r^k \right) - \left(\sum_{k=0}^{n} r^{k+1} \right) \right)$$

Geometric sums

$$\sum_{k=0}^{n} r^k = \frac{1}{1-r}\left(\left(\sum_{k=0}^{n} r^k\right) - \left(\sum_{k=0}^{n} r^{k+1}\right)\right)$$

- What is $\sum_{k=0}^{n} r^k - \sum_{k=0}^{n} r^{k+1}$?

$$= 1 + r + r^2 + r^3 + \cdots + r^{n-3} + r^{n-2} + r^{n-1} + r^n$$
$$- \left(r + r^2 + r^3 + \cdots + r^{n-3} + r^{n-2} + r^{n-1} + r^n + r^{n+1}\right)$$
$$= 1 - r^{n+1}$$

- Thus, $\sum_{k=0}^{n} r^k = \frac{1}{1-r}\left(\left(\sum_{k=0}^{n} r^k\right) - \left(\sum_{k=0}^{n} r^{k+1}\right)\right)$

$$= \frac{1}{1-r}\left(1 - r^{n+1}\right)$$
$$= \frac{1 - r^{n+1}}{1-r}$$

Infinite sequences

- An infinite sequence is a sequence that does not end:

$$a_0, a_1, a_2, a_3, a_4, a_5, \ldots$$
$$a, a+d, a+2d, a+3d, a+4d, a+5d, \ldots$$
$$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots$$

- This is useful for engineers:
  - Once you turn a system on,

    you don't know when it will be turned off
- A *series* is a sum of an infinite sequence:

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$
$$a + (a+d) + (a+2d) + (a+3d) + (a+4d) + (a+5d) + \cdots$$
$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots$$

Arithmetic series

- An arithmetic series is boring:

$$\sum_{k=0}^{\infty}(a+kd)=\begin{cases}\infty & \text{if } d>0 \text{ or } (d=0 \text{ and } a>0)\\ 0 & \text{if } d=0 \text{ and } a=0\\ -\infty & \text{if } d<0 \text{ or } (d=0 \text{ and } a<0)\end{cases}$$

Geometric series

- What about geometric series $\sum_{k=0}^{\infty} ar^k$?
  - Recall that $\sum_{k=0}^{n} r^k = \dfrac{1-r^{n+1}}{1-r}$
  - So if $|r|<1$, we note that as n becomes larger and larger, $r^{n+1} \to 0$
  - Thus, $\sum_{k=0}^{\infty} r^k = \dfrac{1-\cancel{r^{\infty+1}}}{1-r} = \dfrac{1}{1-r}$

Geometric series

- For example,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2$$

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}=1.5$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots=\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$$

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\cdots=\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k=\frac{1}{1-\left(-\frac{1}{3}\right)}=\frac{1}{\frac{4}{3}}=\frac{3}{4}$$

Take-aways

- What you must remember from this talk includes:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$\sum_{k=0}^{n} 1 = n+1$$

$$\sum_{k=0}^{n} ba_k = b\sum_{k=0}^{n} a_k$$

$$\sum_{k=0}^{n} (a_k + b_k) = \left(\sum_{k=0}^{n} a_k\right) + \left(\sum_{k=0}^{n} b_k\right)$$

Summary

- In this topic, we've introduced sequences, sums and series
  - A finite sequence is a fixed number of values in an order
  - A sum of a finite sequence is the sum of those terms
  - We described arithmetic and geometric sequences
  - We found the sum of these sequences
  - We found the formula for an infinite geometric sum, also known as the geometric series
  - Sometimes a finite sum is called a finite series

References

[1] https://en.wikipedia.org/wiki/Sequence

[2] https://en.wikipedia.org/wiki/Arithmetic_progression

[3] https://en.wikipedia.org/wiki/Geometric_progression

Acknowledgments

None so far.

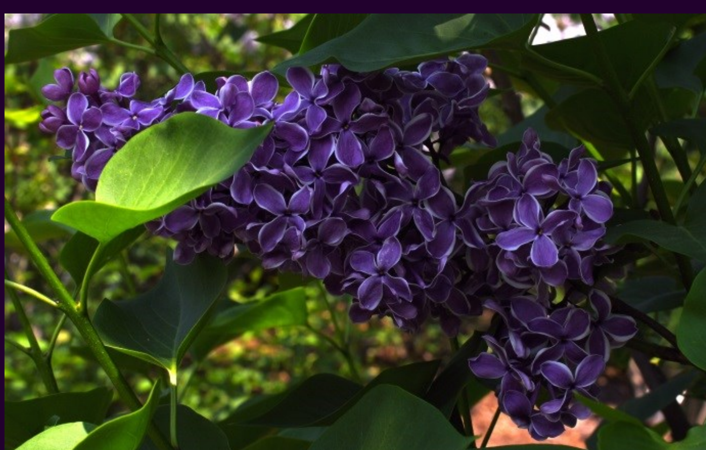
Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.



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