1.6.13 Polynomials with real coefficients
Outline

• In this topic, we will
  – Examine the properties of the roots of polynomials with real coefficients
  – Look at some examples
  – Emphasize that this is one of the most important results in this course
Polynomials with real coefficients

- A polynomial with real coefficients is any polynomial where all coefficients are real.
  - For example, 
    \[ z^5 + 4.2z^4 + 2.7z^3 + 9.3z^2 - 0.8z + 1.2 \]
  - Any such polynomial of degree \( n \) may be written as
    \[
    p(z) = \sum_{k=0}^{n} \alpha_k z^k \\
    = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_2 z^2 + \alpha_1 z + \alpha_0
    \]
  - Important: the coefficient of \( z^k \) is \( \alpha_k \)
Properties of the complex conjugate

- Let us review the properties of the complex conjugate:
  - For any complex $z$: $(z^*)^* = z$
  - For a real number $\alpha$, $\alpha^* = \alpha$
  - For the sum of two complex numbers, $(z_1 + z_2)^* = z_1^* + z_2^*$
  - Consequently, $\left(\sum_{k=1}^{n} z_k\right)^* = \sum_{k=1}^{n} z_k^*$
  - For the product of two complex numbers, $(z_1z_2)^* = z_1^*z_2^*$
  - For a complex number raised to an integer exponent, $(z^n)^* = (z^*)^n$
Polynomials with real coefficients

- Suppose that $r$ is a root of a complex polynomial with real coefficients:

  $$p(r) = \sum_{k=0}^{n} \alpha_k r^k = 0$$

- We have two possibilities: either $r$ is real, or $r$ is not real
  - Let us suppose that $r$ is not real
Polynomials with real coefficients

• If $r$ is not real, let us conjugate both sides of $p(r) = \sum_{k=0}^{n} \alpha_k r^k = 0$

\[
\left( \sum_{k=0}^{n} \alpha_k r^k \right)^* = 0^*
\]

\[
\sum_{k=0}^{n} \left( \alpha_k r^k \right)^* = 0
\]

\[
\sum_{k=0}^{n} \alpha_k^* (r^*)^k = 0
\]

But each $\alpha_k^* = \alpha_k$

\[
\sum_{k=0}^{n} \alpha_k (r^*)^k = 0
\]

• Is this not $p(r^*) = \sum_{k=0}^{n} \alpha_k (r^*)^k$?

– Consequently, if $r$ is a root, the $r^*$ is also a root
Polynomials with real coefficients

Theorem:

If \( p \) is a polynomial with real coefficients, and \( r \) is a root, then so is \( r^* \).

Proof: Assume \( p(r) = \sum_{k=0}^{n} \alpha_k r^k = 0 \).

\[
p(r^*) = \sum_{k=0}^{n} \alpha_k (r^*)^k = \sum_{k=0}^{n} \alpha_k (r^k)^* = \sum_{k=0}^{n} \alpha_k^* (r^k)^* = \left( \sum_{k=0}^{n} \alpha_k r^k \right)^* = 0^* = 0 \]

But each \( \alpha_k^* = \alpha_k \).
To reiterate:

- If $p$ is a polynomial with real coefficients and $r$ is a non-real complex root of $p$ then $r^*$ is also a root of $p$
- We will describe $r$ and $r^*$ as a complex-conjugate pair
- More generally, if $p$ is a polynomial with real coefficients and $r$ is a non-real complex root with multiplicity $m$, then $r^*$ is also a root of $p$, also of multiplicity $m$
Polynomials with non-real coefficients

• Note that if the coefficients are not real, then the correct statement is

If $r$ is a root of the polynomial $p$, then $r^*$ is a root of the polynomial with the conjugate of the coefficients

• For example, $r = 2 - j$ is a root of $z^2 + (1 - 4j)z - 1 + 13j$
  - Therefore, $r^* = 2 + j$ is a root of $z^2 + (1 + 4j)z - 1 - 13j$
  - Evaluating the first polynomial at $r^* = 2 + j$ results in $8 + 10j \neq 0$
Examples

• To demonstrate,
  – We already know the roots of $z^2 + 1$ are $j$ and $-j$
  – The roots of $z^2 + bz + c$ are

$$-\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

• If $b^2 - 4c \geq 0$, then both roots are real
• If $b^2 - 4c < 0$, then the roots are

$$-\frac{b}{2} \pm \frac{\sqrt{4c - b^2}}{2}j$$
Examples

- To demonstrate,
  - Consider the polynomial
    \[ z^5 + 3z^4 + 7z^3 + 5z^2 + 2z + 1 \]
  - Its roots are:
    \[
    -0.035181403011327564 + 0.51437834108030281j \\
    -0.035181403011327564 - 0.51437834108030281j \\
    -1.0777404719073206 + 1.9229758521534597j \\
    -1.0777404719073206 - 1.9229758521534597j \\
    -0.77415625016270363 
    \]
Examples

• To demonstrate,
  – Consider the polynomial

  \[ z^5 + 4z^4 + 7z^3 + 12z^2 + 10z + 2 \]

• Its roots are:
  – \(-0.28217940701111523\)
  – \(-1\)
  – \(-2.5426012611094618\)
  – \(-0.087609665939711494 + 1.6673028066114304j\)
  – \(-0.087609665939711494 - 1.6673028066114304j\)
Summary

• In this topic, you now
  – Understand the significance of the roots of a polynomial with real coefficients
    • Their roots are either real or come in complex conjugate pairs
  – Have been exposed to the proof
  – Are aware of some examples of this result
  – Should be aware that this will be one of the most significant results in this course
References

None so far.
Colophon

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