UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical & Computer Engineering

NE 112 *Linear algebra for nanotechnology engineering*

1.6.8.3 Subtraction of complex numbers

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Introduction

- In this topic, we will
  - Define the subtraction of complex numbers
  - Look at a geometric interpretation
  - Make some observations
  - Use this to define a distance between complex numbers

Complex subtraction

- We define $z - w$ as $z + (-w)$ or $z + (-1)w$
- If $z = \alpha + \beta j$ and $w = \gamma + \delta j$, we have

$$\begin{aligned} z - w &= (\alpha + \beta j) - (\gamma + \delta j) \\ &= (\alpha + \beta j) + (-\gamma - \delta j) \\ &= (\alpha - \gamma) + (\beta - \delta) j \end{aligned}$$

Complex subtraction

- For example, if $z = 0.7 - 8.4j$ and $w = 1.5 + 6.3j$, it follows that

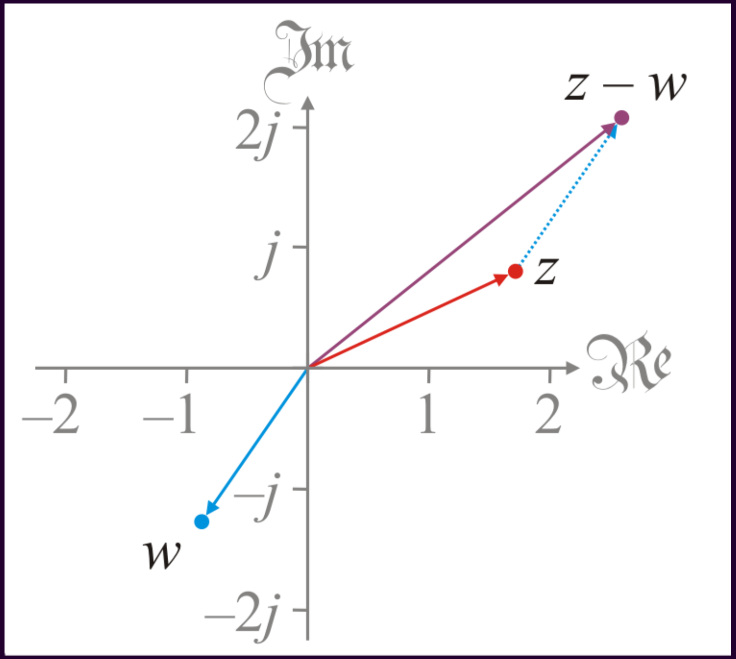
$$\begin{aligned} z - w &= (0.7 - 8.4j) - (1.5 + 6.3j) \\ &= (0.7 - 1.5) + (-8.4 - 6.3)j \\ &= -0.8 - 14.7j \end{aligned}$$

- If $z = -9.6 + 0.1j$ and $w = -7.5 - 8.2j$, and again we see

$$\begin{aligned} z - w &= (-9.6 + 0.1j) - (-7.5 - 8.2j) \\ &= (-9.6 + 7.5) + (0.1 + 8.2)j \\ &= -2.1 + 8.3j \end{aligned}$$

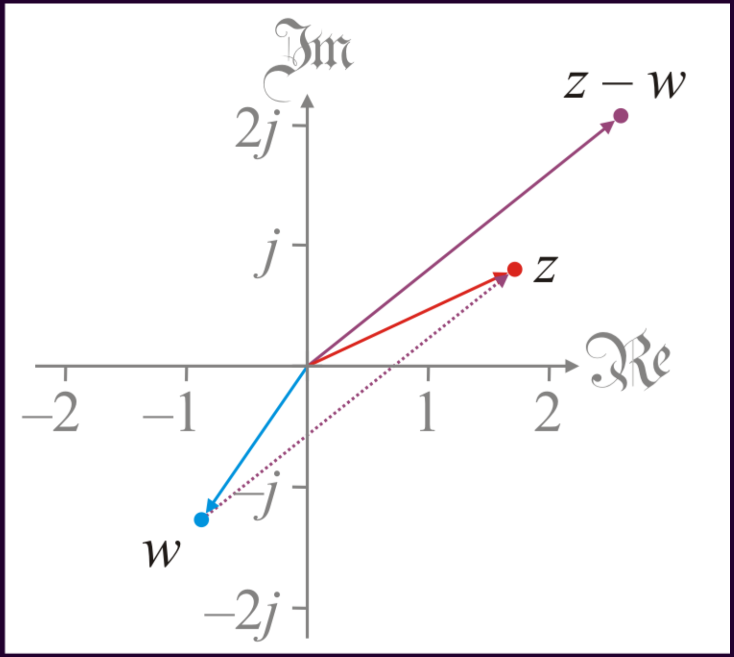
Complex subtraction

- Going back to our geometric interpretation:
  - Here are z and w
  - The value $-w$ is the reflection of w through 0
  - Add $-w$ to z
  - This equals $z - w$



Complex subtraction

- Visually,
  - The value $z - w$ is a complex number going from w to z



Observation

Theorem: $z - w = -(w - z)$

- That is, $w - z$ is the additive inverse of $z - w$

Proof: $$\begin{aligned}(z-w)+(w-z) &= z+(-w)+w+(-z)\\ &= z+(-z)+w+(-w)\\ &= 0+0=0 \quad \mathcal{QED}\end{aligned}$$

Theorem: Complex subtraction is not commutative

Proof: If $z = 1$ and $w = j$,

$$z - w = 1 - j \neq -1 + j = w - z \quad \mathcal{QED}$$

Observation

- By the way, you already use this when subtracting real numbers
  - Calculate $x - y$ when $x > y$:

$$\begin{array}{r} 8.00187 \\ -4.84459 \\ \hline 3.15728 \end{array}$$

  - How do you calculate $x - y$ when $y > x$?

$$\begin{array}{r} 0.56869 \\ -4.27552 \\ \hline ?.????? \end{array}$$

  - Calculate $y - x$ and negate the result, or $x - y = -(y - x)$

$$\begin{array}{r} 4.27552 \\ -0.56869 \\ \hline 3.70683 \end{array} \longrightarrow -3.70683$$

Observation

Theorem: $z - w = 0$ if and only if $z = w$

- That is, $z - w$ is not 0 if z and w are different complex numbers

Proof: Given $z - w = 0$, add w to both sides:

$$(z + (-w)) + w = w$$

$$z + ((-w) + w) = w$$

$$z + 0 = w$$

$$z = w \quad \textit{QED}$$

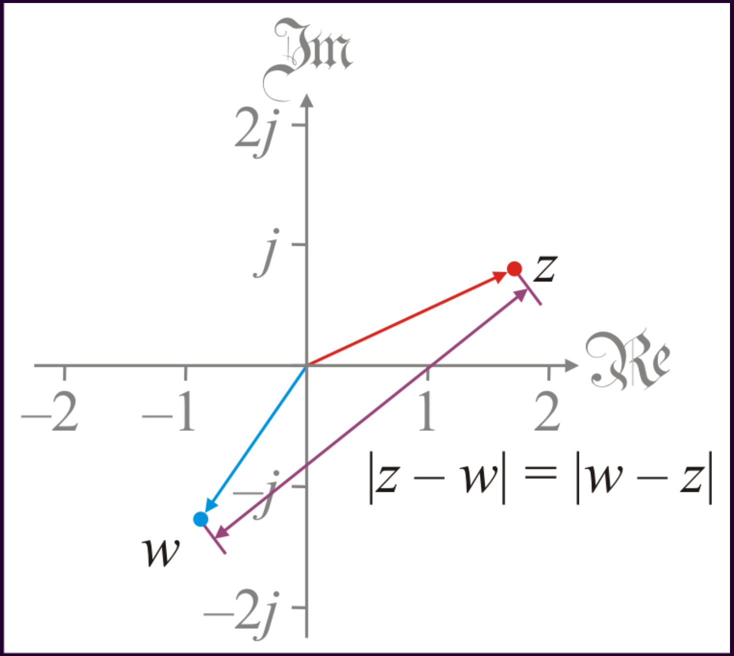
Distance between complex numbers

- We can also interpret $|z - w|$ to be the distance between two complex numbers

- Properties for you to prove:
  1. $|z - w| \geq 0$ — The distance is non-negative
  2. $|z - w| = |w - z|$ — The distance from z to w equals the distance from w to z
  3. $|z - w| = 0$ if and only if $z = w$
  4. $|\gamma z - \gamma w| = |\gamma||z - w|$ — Multiplying both z and w by a real number γ scales the distance by $|\gamma|$

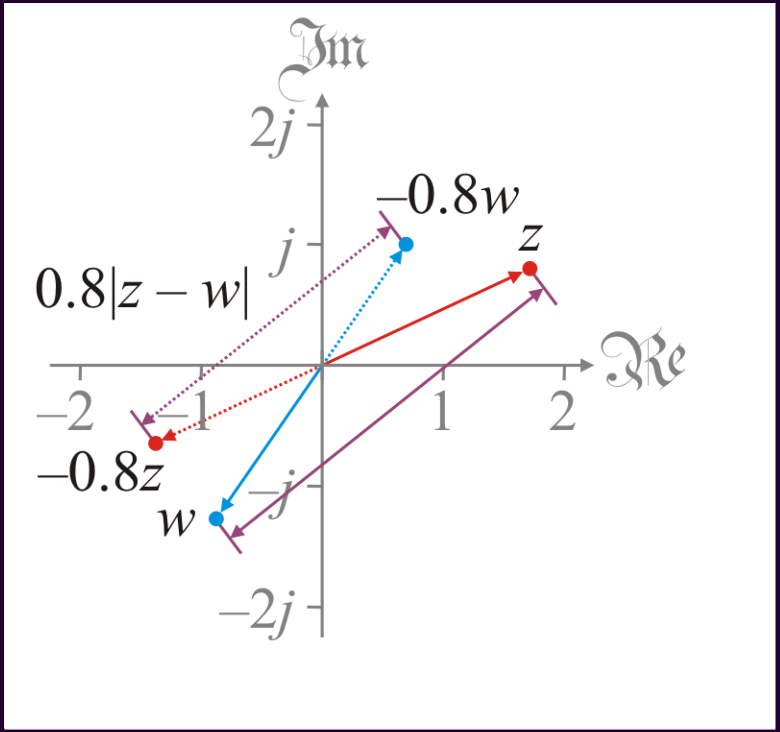
Distance between complex numbers

- In our geometric interpretation of complex numbers, we see that $|z - w|$ is reasonably interpreted as the distance



Distance between complex numbers

- Let's also look the property $|\gamma z - \gamma w| = |\gamma||z - w|$ visually:
  - Here are z and w with the distance $|z - w|$
  - Here are $2z$ and $2w$ with the distance $2|z - w|$
  - Here are $-0.8z$ and $-0.8w$ with the distance $0.8|z - w|$



Summary

- In this topic, we've introduced the subtraction of complex numbers
  - We defined $z - w$ so that $z + (-w) = z + (-1)w$
  - Observed that subtraction is not commutative
  - We looked at a few other properties
  - Interpreted $|z - w|$ as the distance between z and w

References

[1] https://en.wikipedia.org/wiki/Complex_number#Addition_and_subtraction
[2] https://en.wikipedia.org/wiki/Absolute_value#Complex_numbers

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

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