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NE 112 *Linear algebra for nanotechnology engineering*

10 Matrix decompositions

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Introduction

- In this topic, we will
  - Review prime and polynomial factorizations
  - Learn that the word decomposition is often synonymous with factorization
  - Introduce the concept of a matrix decomposition
  - Describe five matrix decompositions useful in engineering
  - Identify those decompositions we will see in this course

Decompositions and factorizations

- In secondary school, you have heard the word *factorization*
  - This is the process of breaking down a more complex structure into simpler structures
  - Another term for this process is *decomposing*
  - We will discuss *matrix decompositions*, but this process is also known as *matrix factorization*

Prime factorization

- In secondary school, you have already learned of factorizations or decompositions:
  - Every positive integer has a unique prime factorization

$$91 = 7 \times 13$$

$$1970 = 2 \times 5 \times 197$$

$$151308335081308941533 4 = 2 \times 3 \times 13^2 \times 137 \times 19249 \times 20411 \times 27722467$$

  - Many public-key cryptosystems rely on the difficulty of factoring integers to send secure messages

$$103665097958289904342438570560669042336391651299123031549302027602242813630593634896235339$$
$$= 2528515060866082428574792897685160982506 31 \times 4099841031707436283465163995252004344396874 27869$$

Polynomial factorization

- In secondary school, you saw that all polynomials could be written as a factorization of either monic linear or quadratic polynomials multiplied by a scaling factor

$$3x^7 + 33x^6 + 111x^5 + 276x^4 + 621x^3 + 363x^2 + 1533x - 2940$$
$$= 3(x - 1)(x + 4)(x + 7)(x^2 - x + 5)(x^2 + 2x + 7)$$

- In this course, we have taken this one step further:

$$3x^7 + 33x^6 + 111x^5 + 276x^4 + 621x^3 + 363x^2 + 1533x - 2940$$
$$= 3(x-1)(x+4)(x+7)\left(x+1+\sqrt{6}j\right)\left(x+1-\sqrt{6}j\right)\left(x-\frac{1}{2}+\frac{\sqrt{19}}{2}j\right)\left(x-\frac{1}{2}-\frac{\sqrt{19}}{2}j\right)$$

Matrix factorizations or decompositions

- A matrix is representation of a linear mapping or linear operator, and it is difficult to understand

$$A = \begin{pmatrix} 3 & 2 & 1 & -3 \\ 2 & -3 & 2 & -2 \\ -1 & 0 & 4 & 1 \\ 1 & 2 & -1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 & 1 & -3 \\ 2 & -3 & 2 & -2 \\ -2 & 0 & 4 & 1 \\ 1 & 2 & -1 & -1 \end{pmatrix}$$

  – A factorization or decomposition of a matrix tries to rewrite the matrix as a product of matrices that provide information about the original matrix

Looking ahead

- In this course, we will see two, maybe three, matrix decompositions
  - In this chapter, we will see the PLU decomposition
    - This is usually just called an LU decomposition
  - Later, we will see:
    - The eigenvalue decomposition or *diagonalization*
    - The singular-value decomposition (SVD)

- Not all matrices may be so decomposed
  - We will see that diagonalization really only applies to those matrices where the matrix equals its transpose: $A = A^{\mathrm{T}}$

Examples

- We will quickly describe the following:
  - LU decomposition
  - Cholesky decomposition
  - QR decomposition
  - Diagonalization (or eigenvalue decomposition)
  - Singular-value decomposition

PLU decomposition

- Every $m \times n$ matrix A can be written as the product $A = PLU$ where:
  - P is an $m \times m$ permutation matrix
  - L is a lower-triangular $m \times m$ matrix with all ones on the diagonal
  - U is an upper triangular $m \times n$ matrix

$$\begin{pmatrix} -1.5 & 2.1 & 6.4 & 1.7 \\ 2 & 3.9 & 3.1 & 11.2 \\ 5 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.4 & 0.9 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

Cholesky decomposition

- Every real $n \times n$ matrix A where $A = A^{\mathrm{T}}$
  and the angle between $\mathbf{u}$ and $A\mathbf{u}$ is less than 90° for all $\mathbf{u}$

  can be written as the product

$$A = LL^{\mathrm{T}}$$

where:

- L is a lower-triangular $n \times n$ matrix with all non-zero positive real numbers on the diagonal

$$\begin{pmatrix} 0.09 & -0.06 & 0.03 & -0.18 \\ -0.06 & 0.2 & 0.18 & -0.6 \\ 0.03 & 0.18 & 0.9 & 0.54 \\ -0.18 & 0.6 & 0.54 & 7.09 \end{pmatrix} = \begin{pmatrix} 0.3 & 0 & 0 & 0 \\ -0.2 & 0.4 & 0 & 0 \\ 0.1 & 0.5 & 0.8 & 0 \\ -0.6 & 1.2 & 0 & 2.3 \end{pmatrix} \begin{pmatrix} 0.3 & -0.2 & 0.1 & -0.6 \\ 0 & 0.4 & 0.5 & 1.2 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 2.3 \end{pmatrix}$$

QR decomposition

- Every real $m \times n$ matrix A can be written as the product
$$A = QR$$
where:
  - Q is an orthogonal $m \times m$ matrix
  - R is an upper-triangular $m \times n$ matrix

$$\begin{pmatrix} 1.92 & 2.3 & 0.14 & 4.06 \\ -2.56 & 1.6 & 0.48 & -2.58 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} -3.2 & -0.1 & 0.3 & -4.5 \\ 0 & 2.8 & 0.4 & 1.7 \end{pmatrix}$$

Diagonalization

- Every real $n \times n$ matrix A where $A = A^{\mathrm{T}}$ can be written as

$$A = UDU^{\mathrm{T}}$$

where:

  - U is an orthogonal $n \times n$ matrix
  - D is a diagonal $n \times n$ matrix with all real entries on that diagonal

$$\begin{pmatrix} 1.569 & -0.405 & 0.595 & -0.817 \\ -0.405 & 0.633 & -1.519 & -1.595 \\ 0.595 & -1.519 & -0.183 & -1.155 \\ -0.817 & -1.595 & -1.155 & -1.119 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1 & 0.5 & 0.5 & 0.7 \\ 0.5 & -0.1 & 0.7 & -0.5 \\ 0.5 & -0.7 & -0.1 & 0.5 \\ 0.7 & 0.5 & -0.5 & -0.1 \end{pmatrix} \begin{pmatrix} -3.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1.7 & 0 \\ 0 & 0 & 0 & 2.4 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.5 & 0.7 \\ 0.5 & -0.1 & -0.7 & 0.5 \\ 0.5 & 0.7 & -0.1 & -0.5 \\ 0.7 & -0.5 & 0.5 & -0.1 \end{pmatrix}$$

Singular-value decomposition (SVD)

- Every real $m \times n$ matrix A can be written as the product

$$A = U\Sigma V^{\mathrm{T}}$$

  where:
  - U is an orthogonal $m \times m$ matrix
  - Σ is a diagonal $m \times n$ matrix with all real entries on that diagonal
  - V is an orthogonal $n \times n$ matrix

$$\begin{pmatrix} -1.382 & -0.254 & 1.282 & -1.994 \\ 0.824 & -0.872 & -2.024 & 0.008 \end{pmatrix}$$

$$= \begin{pmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{pmatrix} \begin{pmatrix} 1.7 & 0 & 0 & 0 \\ 0 & 3.2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.5 & 0.7 \\ 0.5 & -0.1 & -0.7 & 0.5 \\ 0.5 & 0.7 & -0.1 & -0.5 \\ 0.7 & -0.5 & 0.5 & -0.1 \end{pmatrix}$$

Looking ahead

- You do not have to know any of the matrix decompositions we described here but will not cover
  - This is just making you aware of the variety of different decompositions that are useful in different circumstances

- In this topic, we will only look at the LU decomposition
  - In future topics, we will look at the eigenvalue and singular-value decompositions

Summary

- Following this topic, you now
  - Have reviewed the decompositions or factorizations from secondary school
    - Prime and polynomial factorizations
  - Are aware of five useful matrix decompositions or matrix factorizations that you may come across in the future

References

[1] https://en.wikipedia.org/wiki/Matrix_decomposition
[2] https://en.wikipedia.org/wiki/LU_decomposition
[3] https://en.wikipedia.org/wiki/Cholesky_decomposition
[4] https://en.wikipedia.org/wiki/QR_decomposition
[5] https://en.wikipedia.org/wiki/Diagonalizable_matrix
[6] https://en.wikipedia.org/wiki/Singular_value_decomposition

Acknowledgments

None so far.

Colophon

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