UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
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NE 112 *Linear algebra for nanotechnology engineering*

10.1 The PLU decomposition

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Introduction

- In this topic, we will
  - Review row operations as matrices and Gaussian elimination with partial pivoting
  - Work through an example of performing Gaussian elimination with partial pivoting but using row-operation matrices
  - Observe that we should be able to write any matrix as a product of three matrices PLU
  - See how this can speed up solving a system of linear equations when the target vector changes but the matrix remains constant
  - Describe an algorithm for finding P^{T}, L and U and work through an example

Naming

- This algorithm is normally called the *LU* decomposition
  - In this course, I describe this as the *PLU* decomposition
  - This emphasizes that the matrix is decomposed into three separate matrices

PLU decomposition

- The *PLU* decomposition says that every $m \times n$ matrix A can be written as the product $A = PLU$ where:
  - P is an $m \times m$ permutation matrix
  - L is a lower-triangular $m \times m$ matrix with all ones on the diagonal
  - U is an upper triangular $m \times n$ matrix
- We will use a constructive proof:
  - We will begin with an arbitrary matrix and demonstrate that we can determine these three matrices
- We will use Gaussian elimination with partial pivoting to find these three matrices
  - The partial pivoting (swapping rows) produces the P matrix
  - If you ignore partial pivoting, the P matrix will be the identity matrix, but the result will generally be less accurate

Review of Gaussian elimination

- Recall that with Gaussian elimination
  - The goal was to find a row-equivalent matrix that is in row-echelon form
- To find this, we proceeded column-by-column where we:
  - Determine if we have to swap rows,
    and if so, we did
  - We then add appropriate multiples of one row onto rows below it

Row operations as matrices

- Recall also that each row operation can be represented by a row-operation matrix

$$R_{R2\leftrightarrow R5}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad R_{-0.7\cdot R1\to R4}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Recall also that we could find the inverses of these matrices:

$$R_{R2\leftrightarrow R5}^{-1} = R_{R2\leftrightarrow R5} \qquad R_{-0.7\cdot R1\to R4}^{-1} = R_{0.7\cdot R1\to R4}$$

Row operations as matrices

- Recall that a matrix that is in row-echelon form must also be an upper-triangular matrix
  - We applied a sequence of row operations to transform that matrix into one that is in row-echelon form
  - Each row operation could also be performed by multiplying on the left by a row-operation matrix

Row operations as matrices

- For example, suppose we had this matrix

$$\begin{pmatrix} -1.5 & 2.1 & 6.4 & 1.7 \\ 2 & 3.9 & 3.1 & 11.2 \\ 5 & 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1.5 & 2.1 & 6.4 & 1.7 \\ 2 & 3.9 & 3.1 & 11.2 \\ 5 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 2 & 3.9 & 3.1 & 11.2 \\ -1.5 & 2.1 & 6.4 & 1.7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -0.4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1.5 & 2.1 & 6.4 & 1.7 \\ 2 & 3.9 & 3.1 & 11.2 \\ 5 & 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 2.7 & 2.3 & 10.8 \\ 0 & 3 & 7 & 2 \end{pmatrix}$$

Row operations as matrices

- We would proceed by swapping Row 2 and Row 3 and then adding –0.9 times Row 2 onto Row 3

$$\begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 2.7 & 2.3 & 10.8 \\ 0 & 3 & 7 & 2 \end{pmatrix}$$

  - Thus, we would have:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.9 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -0.4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}A$$

$$= \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

Swapping row-operation matrices

- Thus we have the following sequence of matrix multiplications
  - You can also show we can swap two such matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-0.9&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0.3&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-0.4&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}A$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-0.9&1\end{pmatrix}\begin{pmatrix}1&0&0\\0.3&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\-0.4&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}A$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-0.9&1\end{pmatrix}\begin{pmatrix}1&0&0\\0.3&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-0.4&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}A$$

Swapping row-operation matrices

- Consequently,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.9 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} A$$

$$= \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

Multiplying by an inverse

- Now, suppose we have the following:

$$AB = C$$

- Suppose we know that A is invertible,
  in which case, we can multiply both sides by A^{-1}:

$$A^{-1}AB = A^{-1}C$$

- However, $A^{-1}A$ simplifies to I and $IB = B$, so we have

$$B = A^{-1}C$$

Multiplying by an inverse

- Starting on the left, we can multiply both sides by the inverses of each of the square matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.9 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}A$$

$$= \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.9 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.9 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

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$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}A$$

$$= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0.4 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -0.3 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0.9 & 1\end{pmatrix}\begin{pmatrix}5 & 3 & 2 & 1\\ 0 & 3 & 7 & 2\\ 0 & 0 & -4 & 9\end{pmatrix}$$

Multiplying by an inverse

- Starting on the left, we can multiply both sides by the inverses of each of the square matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.9 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}$$

The *PLU* decomposition

- Thus, we have:

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.9 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.4 & 0.9 & 1 \end{pmatrix}}\underset{\begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}}{\begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix}}$$

The *PLU* decomposition

- Therefore,

$$A = \begin{pmatrix} -1.5 & 2.1 & 6.4 & 1.7 \\ 2 & 3.9 & 3.1 & 11.2 \\ 5 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.4 & 0.9 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 2 & 1 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -4 & 9 \end{pmatrix} = PLU$$

So what?

- Okay, it seems we can rewrite $A = PLU$... so what?
  - Given $\mathbf{v}$, suppose we want to solve $A\mathbf{u} = \mathbf{v}$ for $\mathbf{u}$
  - Normally, we
    - Create the augmented matrix $(A \mid \mathbf{u})$
    - Apply Gaussian elimination with partial pivoting to find a row-equivalent matrix in row-echelon form
    - Use backward substitution to find $\mathbf{u}$
  - Problem: the Gaussian elimination step is computationally very expensive—much more expensive than backward substitution
    - If A is $n \times n$ and there is a unique solution, the Gaussian elimination step would take n times longer to complete than the backward substitution

So what?

- Suppose, instead of solve $A\mathbf{u} = \mathbf{v}$, we instead solve $PLU\mathbf{u} = \mathbf{v}$
  - Step 1: Multiply both sides by the inverse of P
    - The inverse of a permutation matrix is its transpose

$$P^{\mathrm{T}}PLU\mathbf{u} = P^{\mathrm{T}}\mathbf{v}$$

$$LU\mathbf{u} = P^{\mathrm{T}}\mathbf{v}$$

    - Permuting the entries of a vector can be done very fast
  - Let us temporarily assume $U\mathbf{u} = \mathbf{y}$
    - We can solve for $\mathbf{y}$ by using forward substitution with $L\mathbf{y} = P^{\mathrm{T}}\mathbf{v}$
    - Having found $\mathbf{y}$,
      we now solve for $\mathbf{u}$ using backward substitution with $U\mathbf{u} = \mathbf{y}$

So what?

- For example,
  suppose A is a 100 × 100 matrix that describes the response of graphene surface to forces
  - You would like to simulate your design
  - You have a number of inputs: $\mathbf{v}_1, \ldots, \mathbf{v}_{200}$,
    solving $A\mathbf{u}_k = \mathbf{v}_k$ allows you to find the corresponding state
  - This allows you to test your design
- Two possibilities:
  - Use Gaussian elimination each time you solved $(A \mid \mathbf{v}_k)$
  - Calculate $A = PLU$ and then solve $(L \mid P^T \mathbf{v}_k)$ and $(U \mid \mathbf{y}_k)$

- With 200 simulations, the first would take 40 times longer…
  - This would impact your employer's financial bottom line

Example

- For example, suppose we have that

$$A = \begin{pmatrix} -1.5 & 2.1 & 6.4 \\ 2 & 3.9 & 3.1 \\ 5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.4 & 0.9 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 2 \\ 0 & 3 & 7 \\ 0 & 0 & -4 \end{pmatrix} = PLU$$

  - The benefit is harder to see with small matrices, but let's give it a go…

$$A\mathbf{u} = \begin{pmatrix} -8 \\ 7.5 \\ 10 \end{pmatrix} \qquad A\mathbf{u} = \begin{pmatrix} 23.6 \\ -0.1 \\ -2 \end{pmatrix}$$

Example

- With the first example:

$$P^{\mathrm{T}}\begin{pmatrix} -8 \\ 7.5 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -8 \\ 7.5 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ 7.5 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 10 \\ -0.3 & 1 & 0 & -8 \\ 0.4 & 0.9 & 1 & 7.5 \end{array}\right) \qquad \mathbf{y} = \begin{pmatrix} 10 \\ -5 \\ 8 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} 5 & 3 & 2 & 10 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & -4 & 8 \end{array}\right) \qquad \mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

Example

- With the second example:

$$P^{\mathrm{T}}\begin{pmatrix} 23.6 \\ -0.1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 23.6 \\ -0.1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 23.6 \\ -0.1 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & -2 \\ -0.3 & 1 & 0 & 23.6 \\ 0.4 & 0.9 & 1 & -0.1 \end{array}\right) \qquad \mathbf{y} = \begin{pmatrix} -2 \\ 23 \\ -20 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} 5 & 3 & 2 & -2 \\ 0 & 3 & 7 & 23 \\ 0 & 0 & -4 & -20 \end{array}\right) \qquad \mathbf{u} = \begin{pmatrix} 0 \\ -4 \\ 5 \end{pmatrix}$$

PLU decomposition algorithm

- We will now describe an algorithm to find P^{T}, L and U
  - Given an $m \times n$ matrix A, we begin with three *candidate matrices*:

$$\begin{aligned} P^{\mathrm{T}} &\leftarrow I_m \\ L' = L - I_m &\leftarrow O_m \\ U &\leftarrow A \end{aligned}$$

  - Next, apply the steps of Gaussian elimination with the following modifications:
    - Each time you swap two rows,
swap the rows of P^{T}, L', and U
    - Each time you add a multiple α of Row i onto Row j in U,
set the entry (j, i) in L' to $-\alpha$
  - When U is in row-echelon form, we have found U and P^{T} and we assign $L \leftarrow L' + I_m$

PLU decomposition algorithm

- Two points:
  - If you understand the Gaussian elimination algorithm, you can find the three matrices P^{T}, L and U
  - This describes an algorithm
    - You may still wonder whether or not this algorithm actually does find this matrix decomposition
    - For example, why do we only swap the rows of the off-diagonal entries of L?

PLU decomposition algorithm

- To demonstrate this algorithm works,
  we will apply it to the following 4 × 5 matrix

$$A = \begin{pmatrix} 3.2 & -5 & 5.9 & 4.1 & -3 \\ -0.4 & 5.5 & 2.2 & -2.1 & 7.3 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 4 & -1 & 2 & 0 & 3 \end{pmatrix}$$

PLU decomposition algorithm

- We start with our three candidate matrices:

$$P^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 3.2 & -5 & 5.9 & 4.1 & -3 \\ -0.4 & 5.5 & 2.2 & -2.1 & 7.3 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 4 & -1 & 2 & 0 & 3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Swap Row 1 and Row 4 of all three matrices

$$P^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 3.2 & -5 & 5.9 & 4.1 & -3 \\ -0.4 & 5.5 & 2.2 & -2.1 & 7.3 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 4 & -1 & 2 & 0 & 3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Swap Row 1 and Row 4 of all three matrices

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ -0.4 & 5.5 & 2.2 & -2.1 & 7.3 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 3.2 & -5 & 5.9 & 4.1 & -3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Add 0.1 times Row 1 onto Row 2

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ -0.4 & 5.5 & 2.2 & -2.1 & 7.3 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 3.2 & -5 & 5.9 & 4.1 & -3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Add 0.1 times Row 1 onto Row 2

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 3.2 & -5 & 5.9 & 4.1 & -3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Add 0.4 times Row 1 onto Row 3

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 3.2 & -5 & 5.9 & 4.1 & -3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Add 0.4 times Row 1 onto Row 3

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ 0 & 6 & 1 & -3 & 2 \\ 3.2 & -5 & 5.9 & 4.1 & -3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Add –0.8 times Row 1 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ 0 & 6 & 1 & -3 & 2 \\ 3.2 & -5 & 5.9 & 4.1 & -3 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 1:
  - Add –0.8 times Row 1 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & -4.2 & 4.3 & 4.1 & -5.4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 2:
  - Swap Row 2 and Row 3

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & -4.2 & 4.3 & 4.1 & -5.4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 2:
  - Swap Row 2 and Row 3

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ 0 & -4.2 & 4.3 & 4.1 & -5.4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 2:
  - Add –0.9 times Row 2 onto Row 3

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 5.4 & 2.4 & -2.1 & 7.6 \\ 0 & -4.2 & 4.3 & 4.1 & -5.4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 2:
  - Add –0.9 times Row 2 onto Row 3

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ -0.1 & 0.9 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 1.5 & 0.6 & 5.8 \\ 0 & -4.2 & 4.3 & 4.1 & -5.4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 2:
    - Add 0.7 times Row 2 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ -0.1 & 0.9 & 0 & 0 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 1.5 & 0.6 & 5.8 \\ 0 & -4.2 & 4.3 & 4.1 & -5.4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 2:
  - Add 0.7 times Row 2 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ -0.1 & 0.9 & 0 & 0 \\ 0.8 & -0.7 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 1.5 & 0.6 & 5.8 \\ 0 & 0 & 5 & 2 & -4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 3:
  - Swap Row 3 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ -0.1 & 0.9 & 0 & 0 \\ 0.8 & -0.7 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 1.5 & 0.6 & 5.8 \\ 0 & 0 & 5 & 2 & -4 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 3:
  - Swap Row 3 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0.8 & -0.7 & 0 & 0 \\ -0.1 & 0.9 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 1.5 & 0.6 & 5.8 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 3:
  - Add –0.3 times Row 3 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0.8 & -0.7 & 0 & 0 \\ -0.1 & 0.9 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 1.5 & 0.6 & 5.8 \end{pmatrix}$$

PLU decomposition algorithm

- In Column 3:
  - Add –0.3 times Row 3 onto Row 4

$$P^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0.8 & -0.7 & 0 & 0 \\ -0.1 & 0.9 & 0.3 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

PLU decomposition algorithm

- We now add the identity matrix to L' to get L and take the transpose of P^T to get P

$$P^\mathrm{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.4 & 0 & 0 & 0 \\ 0.8 & -0.7 & 0 & 0 \\ -0.1 & 0.9 & 0.3 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

PLU decomposition algorithm

- We now add the identity matrix to L' to get L and take the transpose of P^{T} to get P

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.4 & 1 & 0 & 0 \\ 0.8 & -0.7 & 1 & 0 \\ -0.1 & 0.9 & 0.3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

PLU decomposition algorithm

- We now have that

$$A = \begin{pmatrix} 3.2 & -5 & 5.9 & 4.1 & -3 \\ -0.4 & 5.5 & 2.2 & -2.1 & 7.3 \\ -1.6 & 6.4 & 0.2 & -3 & 0.8 \\ 4 & -1 & 2 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.4 & 1 & 0 & 0 \\ 0.8 & -0.7 & 1 & 0 \\ -0.1 & 0.9 & 0.3 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & 2 & 0 & 3 \\ 0 & 6 & 1 & -3 & 2 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Summary

- Following this topic, you now
  - Have seen a demonstration of how matrix row operations suggest a *PLU* decomposition is possible
  - Understand how to use the decomposition to solve a system of linear equations
  - Are aware that this can help speed up solving a system of equations for many different target vectors when the matrix is fixed
  - Understand the algorithm for finding the matrices P^{T}, L and U

References

[1] https://en.wikipedia.org/wiki/Matrix_decomposition
[2] https://en.wikipedia.org/wiki/LU_decomposition

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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