UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
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NE 112 *Linear algebra for nanotechnology engineering*

3.2 Determining if a subset is a subspace

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Introduction

- In this topic, we will
  - Describe the idea of a subspace
  - Consider a few examples, including in $\mathbf{R}^2$ and $\mathbf{R}^3$
  - Prove a few theorems regarding such subspaces
  - Consider all linear combinations of a finite set of vectors

The zero set

- Given $\mathbf{R}^2$, let us focus on the set containing just the zero vector:

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

- Notice that:
  - If we add any two vectors in S, we get back a vector in S:
    $\mathbf{0} + \mathbf{0} = \mathbf{0}$
  - If we multiply any vector in S by a scalar, we get back a vector in S:
    $\alpha \mathbf{0} = \mathbf{0}$
- Thus, S is closed under vector addition and scalar multiplication
  - Any subset of a vector space that is closed under vector addition and scalar multiplication is said to be a *subspace* of that vector space

The entire vector space

- Another subset of $\mathbf{R}^2$ is all of $\mathbf{R}^2$
  - Given any vector space $\mathcal{V}$, we call both $\{0\}$ and $\mathcal{V}$ the trivial subspaces of $\mathcal{V}$

The empty set

- The empty set is a subset of $\mathbf{R}^2$
  - Is it a subspace?

- Recall that for a set to be a vector space, it must have a zero vector
  - The empty set does not have a zero vector, thus, it is not a subspace

When is a subset a subspace?

- A subset S of a vector space $\mathcal{V}$ is itself a vector space if:
  - The sum of any two vectors in S is again a vector in S
  - All scalar multiples of a vector in S must again be vectors in S

- In this case, S is a vector space
  - Because it is a subset of the vector space $\mathcal{V}$,
    it is called a subspace of the vector space $\mathcal{V}$

When is a subset a subspace?

- Consider the vector space of all polynomials with real coefficients: $\mathbf{R}[x]$
  - Previously, we described how this is a vector space

- Consider the subset S_5 of all polynomials of degree less than or equal to 5
  - Multiplying a polynomial of degree less than or equal to 5 by α must result in a polynomial of degree less than or equal to 5
  - Adding two polynomials of degree less than or equal to 5 must result in a polynomial of degree less than or equal to 5
- Therefore, S_5 must be a vector space,

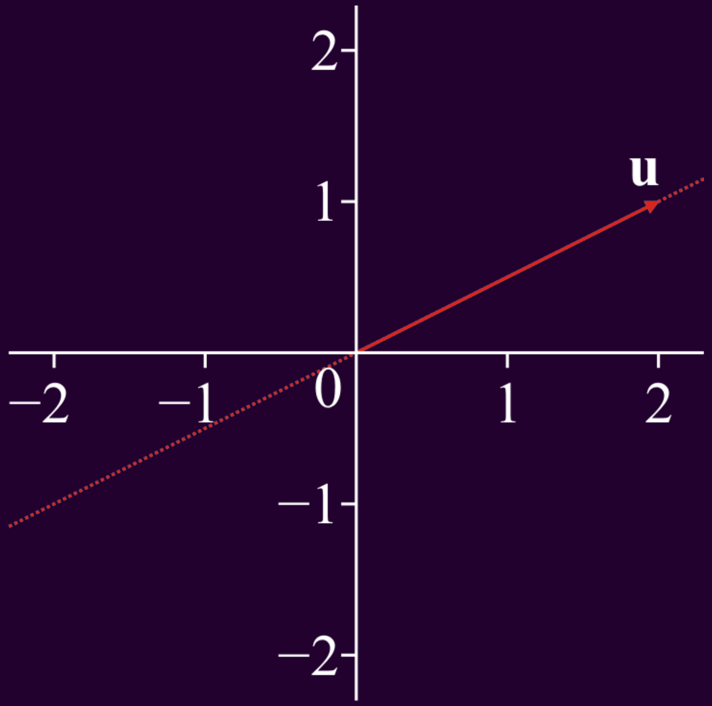
  thus, it is a subspace of $\mathbf{R}[x]$

Discrete signals

- In $\mathbf{R}^2$, consider all scalar multiples of

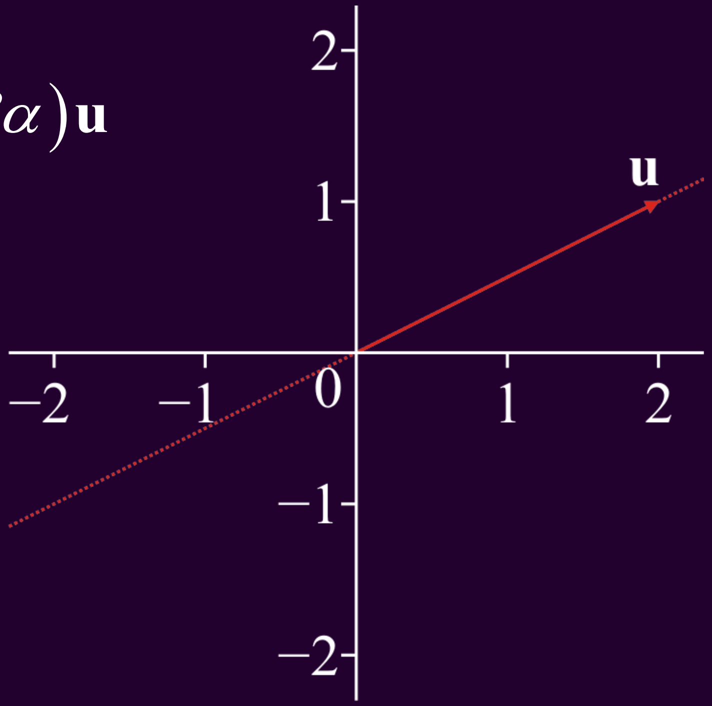
$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

  - This is a line in 2-space passing through the origin



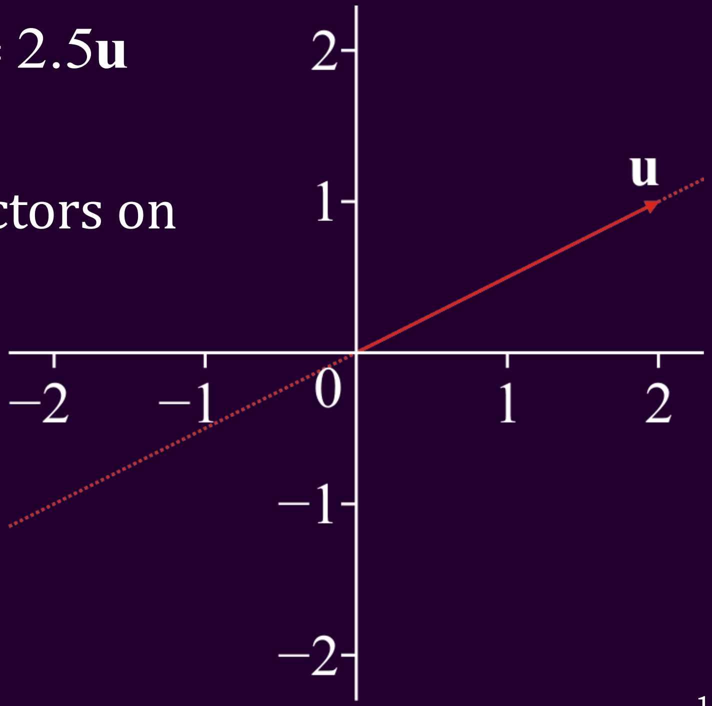
Discrete signals

- For example, $\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ is such a vector
  - We see that $\mathbf{v} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so $\mathbf{v} = \alpha\mathbf{u}$
  - Now, if $\mathbf{v} = \alpha\mathbf{u}$
    note that $\beta\mathbf{v} = \beta(\alpha\mathbf{u}) = (\beta\alpha)\mathbf{u}$



Straight lines in $\mathbf{R}^2$

- For example, $\mathbf{v}_1 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$ are such vectors
- We see that $\mathbf{v}_1 = 3\mathbf{u}$ and $\mathbf{v}_2 = -0.5\mathbf{u}$
  - We also see $\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} 5 \\ 2.5 \end{pmatrix} = 2.5\mathbf{u}$
  - Thus, the sum of these two vectors on this line is still on this line



Straight lines in $\mathbf{R}^2$

Theorem

If $\mathcal{V}$ is any vector space, and $\mathbf{u} \in \mathcal{V}$,

then the subset S of all scalar multiples of $\mathbf{u}$

is a subspace of $\mathcal{V}$.

Proof:

If $\mathbf{v} \in S$, then $\mathbf{v} = \beta\mathbf{u}$ for some β.

Thus, $\alpha\mathbf{v} = \alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$, so $\alpha\mathbf{v} \in S$.

If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 = \beta_1\mathbf{u}$ and $\mathbf{v}_2 = \beta_2\mathbf{u}$ for some β_1 and β_2.

Thus, $\mathbf{v}_1 + \mathbf{v}_2 = \beta_1\mathbf{u} + \beta_2\mathbf{u} = (\beta_1 + \beta_2)\mathbf{u}$, so $\mathbf{v}_1 + \mathbf{v}_2 \in S$.

Therefore, all scalar multiples of $\mathbf{u}$ for a subspace of $\mathcal{V}$. ∎

Straight lines in $\mathbf{R}^2$

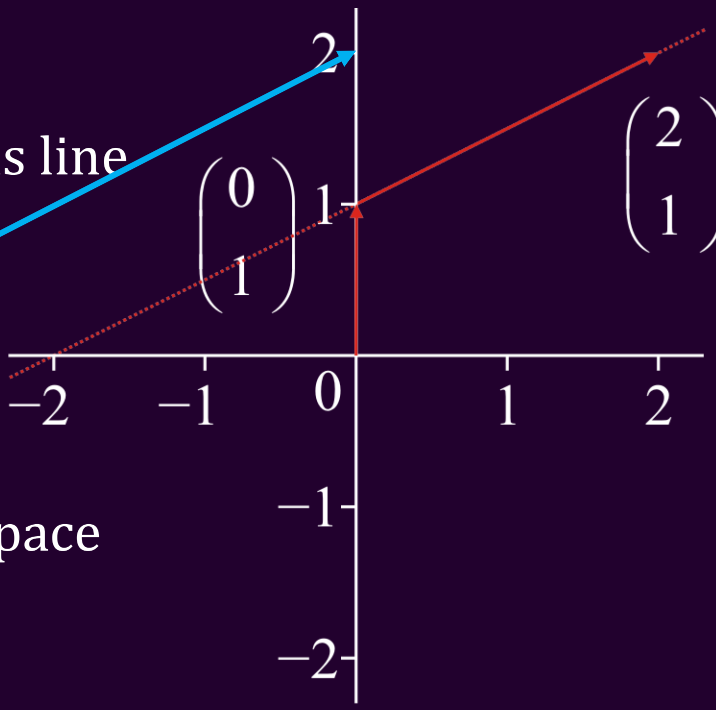
- Thus, any "line through the origin" in any vector space is a subspace:
  - Given $\mathbf{R}[x]$, the vector space of all polynomials
  - Note that $x^2 \in \mathbf{R}[x]$
  - Thus, all scalar multiples αx^2 forms a subspace of $\mathbf{R}[x]$

Straight lines in $\mathbf{R}^2$

- What if the line does not pass through the origin?
  - Consider all vectors of the form

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

  - The vector $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is on this line
  - Problem: $2\mathbf{u} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is not
  - Therefore, this is not a subspace



Subspaces must contain the origin

Theorem

Given a vector space $\mathcal{V}$, any subset $S \subseteq \mathcal{V}$ that does not contain the zero vector is not a subspace of $\mathcal{V}$.

Proof:

If $S = \{\}$, we have already shown it is not a subspace.

If $S \neq \{\}$, then S must have at least one vector $\mathbf{u}$, and by assumption, $\mathbf{u} \neq \mathbf{0}$.

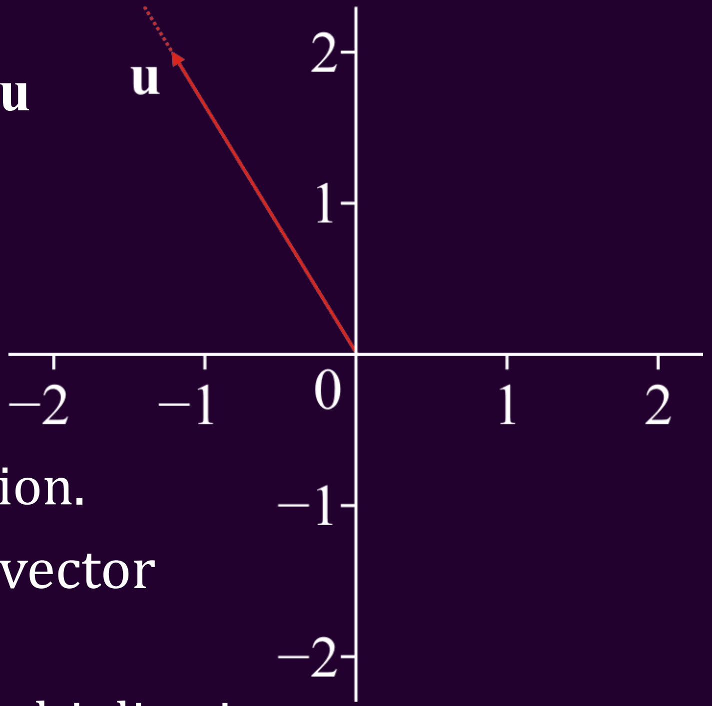
However, $0\mathbf{u} = \mathbf{0}$, and thus,

S is not closed under scalar multiplication.

Therefore S cannot be a subspace. ■

Rays from the origin

- Consider the set T of all non-negative multiples of a vector $\mathbf{u}$
  - This forms a *ray* coming out of the origin



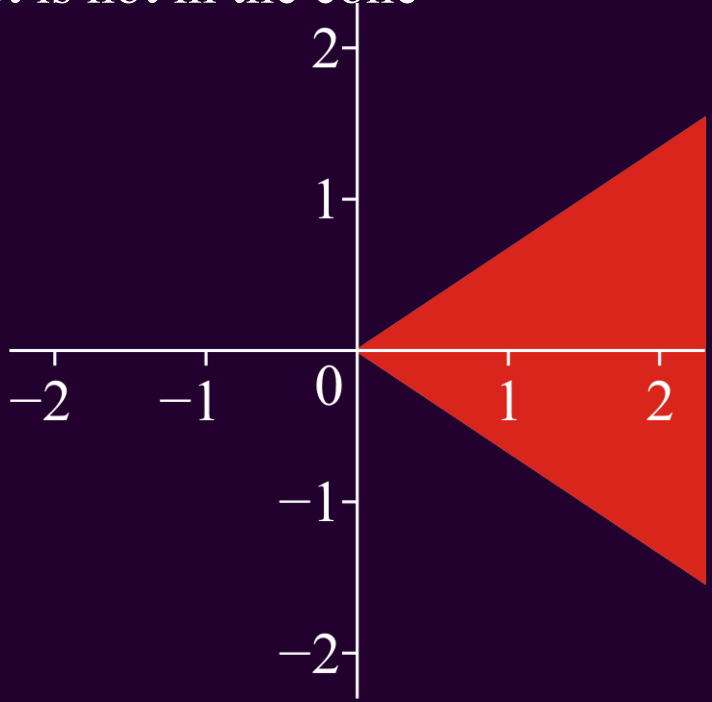
- If $\mathbf{v}_1, \mathbf{v}_2 \in T$, then $\mathbf{v}_1 = \alpha_1 \mathbf{u}$ and $\mathbf{v}_2 = \alpha_2 \mathbf{u}$ for $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$.
  - Therefore,

$$\mathbf{v}_1 + \mathbf{v}_2 = (\alpha_1 + \alpha_2)\mathbf{u}$$

    where $\alpha_1 + \alpha_2 \geq 0$.
  - Thus, T is closed under vector addition.
- However, $-1 \cdot \mathbf{u}$, a scalar multiple of a vector in T is not in T
  - Thus, T is not closed under scalar multiplication.

Cones

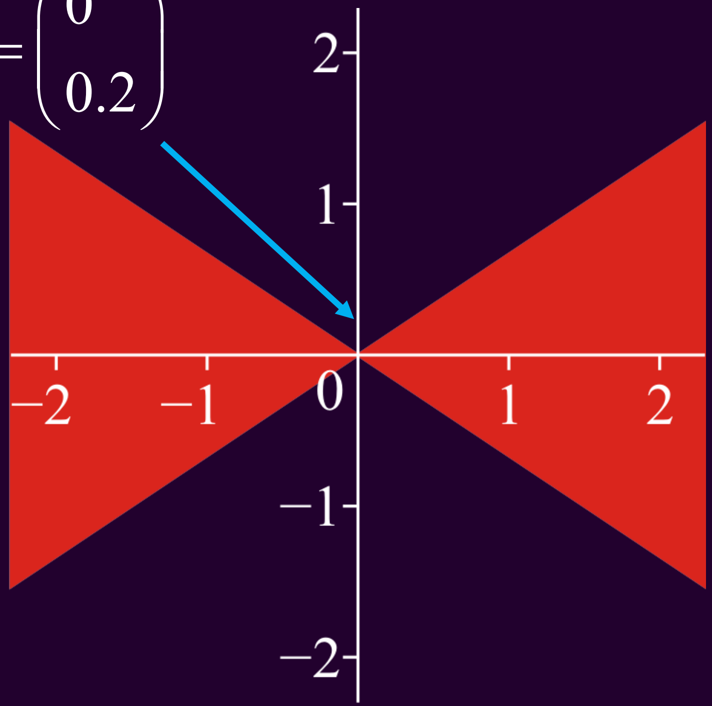
- Is the region in red a subspace of $\mathbf{R}^2$?
  - If we add two vectors in this cone, the sum is still in the cone
  - If we multiply any vector in this cone by –1, apart from **0**, the product is not in the cone



Cones

- Let's add all scalar multiples of all vectors in this cone
  - Is this now a subspace?
    - It is definitely closed under scalar multiplication
    - Unfortunately, try adding:

$$\begin{pmatrix} 1 \\ 0.1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}$$

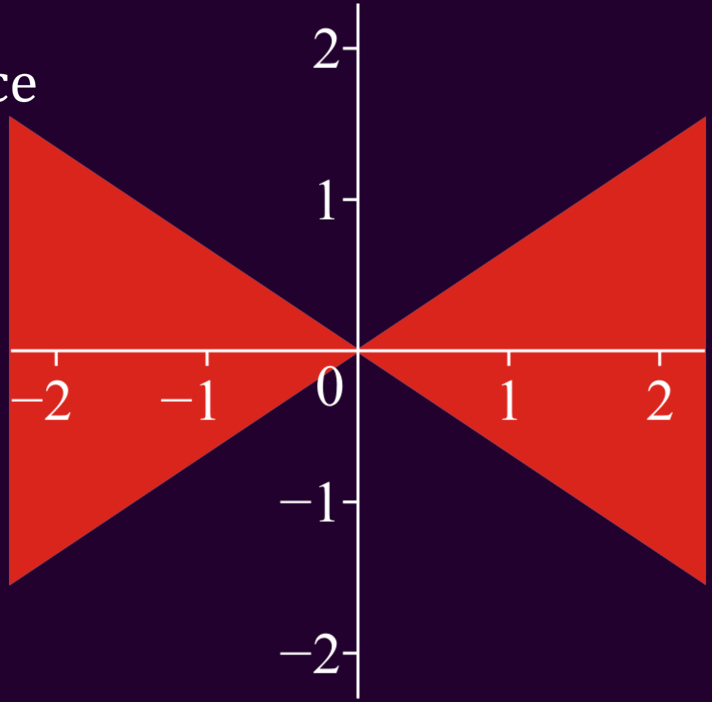


Cones

- Given any vector in $\mathbf{R}^2$, we can find it to be the sum of two vectors in this double cone:

$$\begin{pmatrix}(n+1)x \\ 0\end{pmatrix} + \begin{pmatrix}-nx \\ y\end{pmatrix} = \begin{pmatrix}x \\ y\end{pmatrix}$$

- Thus, the smallest subspace containing this double cone is all of $\mathbf{R}^2$



All subspaces of $\mathbf{R}^2$

- Thus, all subspaces of $\mathbf{R}^2$ include:
  - The trivial subspace $\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$
  - All lines passing through the origin
  - The trivial subspace $\mathbf{R}^2$

Linear combinations

Theorem

Given a finite set of m vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ in some vector space $\mathcal{V}$, the set S of all scalar multiples of these vectors,

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m,$$

forms a subspace.

Proof:

Any vector in this set is of the form $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m$

– Thus, $\beta(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m)$

$$= \beta(\alpha_1\mathbf{u}_1) + \beta(\alpha_2\mathbf{u}_2) + \cdots + \beta(\alpha_m\mathbf{u}_m)$$

$$= (\beta\alpha_1)\mathbf{u}_1 + (\beta\alpha_2)\mathbf{u}_2 + \cdots + (\beta\alpha_m)\mathbf{u}_m$$

– Also, $(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m) + (\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_m\mathbf{u}_m)$

$$= (\alpha_1 + \beta_1)\mathbf{u}_1 + (\alpha_2 + \beta_2)\mathbf{u}_2 + \cdots + (\alpha_m + \beta_m)\mathbf{u}_m$$

Therefore, S is a subspace of $\mathcal{V}$. ■

Linear combinations

- Claim: For all finite-dimensional vector spaces $\mathbf{R}^n$ and $\mathbf{C}^n$, the only subspaces are $\{\mathbf{0}\}$ or all linear combinations of one or more non-zero vectors

$$\left\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\right\}$$

Linear combinations

- For example, in $\mathbf{R}^3$
  - There is the subspace containing the zero vector $\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$
  - Given a non-zero vector **u**,

    all linear combinations of **u** forms a line through the origin

$$\begin{pmatrix}\alpha u_1\\ \alpha u_2\\ \alpha u_3\end{pmatrix}$$

  - If two non-zero vectors are not scalar multiples of each other,

    all linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$ forms a plane through the origin

$$\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \begin{pmatrix}\alpha u_{1,1} + \beta u_{2,1}\\ \alpha u_{1,2} + \beta u_{2,2}\\ \alpha u_{1,3} + \beta u_{2,3}\end{pmatrix}$$

Linear combinations

- For any other description of a subset of a vector space, to show it is a subspace, you must show that:
  - All scalar multiples of one vector that fit that description, must still fit that description
  - The sum of any two vectors that fit that description, must still fit that description

- Note that if the zero vector does not fit the description, you may automatically conclude that the subset is not a subspace
  - We'll look at a few examples next

Summary

- Following this topic, you now
  - Understand the idea of a subspace
  - Know that subspaces must contain the zero vector
  - Are aware that all subspaces of $\mathbf{F}^n$ are the result of taking all linear combinations of a finite set of vectors
    - E.g., all non-trivial subspaces of $\mathbf{R}^3$ in include all lines and planes that pass through the origin
  - Know that to prove a subset is a subspace, you must

    show it is closed under vector addition and scalar multiplication

References

[1] https://en.wikipedia.org/wiki/Linear_subspace

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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