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NE 112 *Linear algebra for nanotechnology engineering*

4.1 The 2-norm for finite-dimensional vectors

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Introduction

- In this topic, we will
  - Define the 2-norm for finite-dimensional vectors $\mathbf{R}^n$ and $\mathbf{C}^n$
  - Define the *distance* between vectors
  - Describe the properties of the 2-norm

Norms of vectors

- If $\mathbf{u} \in \mathbf{F}^n$, we will define the 2-norm of $\mathbf{u}$ as

$$\|\mathbf{u}\|_2 = \sqrt{\sum_{k=1}^{n} |u_k|^2}$$

- Note that if $\mathbf{u} \in \mathbf{R}^n$, then $\|\mathbf{u}\|_2 = \sqrt{\sum_{k=1}^{n} u_k^2}$

- Similarly, if $\mathbf{u} \in \mathbf{C}^n$, then $\|\mathbf{u}\|_2 = \sqrt{\sum_{k=1}^{n} u_k^* u_k}$

Norms of vectors

- For example, if $\mathbf{u} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$, then

$$\begin{aligned} \|\mathbf{u}\|_2 &= \sqrt{3^2 + (-6)^2 + 2^2} \\ &= \sqrt{9 + 36 + 4} \\ &= \sqrt{49} = 7 \end{aligned}$$

- If $\mathbf{u} = \begin{pmatrix} 3 + 2j \\ -1 - 3j \\ -2 + 3j \end{pmatrix}$, then

$$\begin{aligned} \|\mathbf{u}\|_2 &= \sqrt{\left(3^2 + 2^2\right) + \left((-1)^2 + (-3)^2\right) + \left((-2)^2 + 3^2\right)} \\ &= \sqrt{(9 + 4) + (1 + 9) + (4 + 9)} \\ &= \sqrt{36} = 6 \end{aligned}$$

Norms of vectors

- Sometimes, we just write

$$\left\|\begin{pmatrix} 1.8 \\ 6.3 \\ -1.4 \end{pmatrix}\right\|_2 = \sqrt{1.8^2 + 6.3^2 + 1.4^2}$$

$$= \sqrt{3.24 + 39.69 + 1.96}$$

$$= \sqrt{44.89} = 6.7$$

Norms of vectors

- Why the "2-norm"?
  - The "2" refers to each entry being raised to the power 2
- Most of our examples have been with 3-dimensional vectors
  - Nothing restricts this to 2 or 3 dimensions:

$$\left\| \begin{pmatrix} 0.2 \\ -0.4 \\ 1.0 \\ 0.5 \\ -1.2 \end{pmatrix} \right\|_2 = \sqrt{0.04 + 0.16 + 1 + 0.25 + 1.44} = \sqrt{2.89} = 1.7$$

Distance

- We will define the distance between two vectors **u** and **v** as

$$\operatorname{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|_2$$

Properties of norms

- What are the useful properties of 2-norms?
  - First,

$$\|\mathbf{u}\|_2^2 = \sum_{k=1}^{n} |u_k|^2 \geq \sum_{k=1}^{n} 0 = 0$$

Therefore, $\|\mathbf{u}\|_2 \geq \sqrt{0} = 0$

  - Second, $\|\mathbf{0}\|_2^2 = \sum_{k=1}^{n} |0|^2 = \sum_{k=1}^{n} 0 = 0$

and if $\mathbf{u} \neq \mathbf{0}$, then $u_\ell \neq 0$ for some ℓ, so

$$\|\mathbf{u}\|_2^2 = \sum_{k=1}^{n} |u_k|^2 \geq |u_\ell|^2 > 0$$

Therefore, $\|\mathbf{u}\|_2 \geq 0$ and $\|\mathbf{u}\|_2 = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Properties of norms

- What are the useful properties of 2-norms?
  - Third,

$$\|\alpha\mathbf{u}\|_2^2 = \sum_{k=1}^{n}|\alpha u_k|^2$$
$$= \sum_{k=1}^{n}|\alpha|^2|u_k|^2$$
$$= |\alpha|^2\sum_{k=1}^{n}|u_k|^2 \quad = |\alpha|^2\|\mathbf{u}\|_2^2$$

Thus, $\|\alpha\mathbf{u}\|_2 = |\alpha|\|\mathbf{u}\|_2$

Properties of norms

- What are the useful properties of 2-norms?
  - Finally, recall from complex numbers

    The absolute value of the sum of two complex numbers cannot be greater than the sum of the absolute values

    $$|w+z| \leq |w| + |z|$$

  - Similarly, the 2-norm of the sum of two vectors cannot be greater than the sum of the 2-norms of those vectors:

    $$\|\mathbf{u}+\mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$$

Properties of norms

- Thus, the recognized useful properties of the 2-norm include:

$$\|\mathbf{u}\|_2 \geq 0 \text{ and } \|\mathbf{u}\|_2 = 0 \text{ if and only if } \mathbf{u} = \mathbf{0}$$

$$\|\alpha \mathbf{u}\|_2 = |\alpha| \|\mathbf{u}\|_2$$

$$\|\mathbf{u} + \mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$$

Distance

- Going back to the definition of distance:
  - If $\mathbf{u} = \mathbf{v}$, then $\mathbf{u} - \mathbf{v} = \mathbf{0}$,

    Thus $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{0}\|_2 = 0$
  - If $\mathbf{u} \neq \mathbf{v}$, then $u_\ell \neq v_\ell$ for some ℓ, so $u_\ell - v_\ell \neq 0$ for that ℓ

    Therefore, $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$, so $\text{dist}(\mathbf{u}, \mathbf{v}) > 0$
  - Finally, the distance is symmetric:

$$\begin{aligned}\text{dist}(\mathbf{v}, \mathbf{u}) &= \|\mathbf{v} - \mathbf{u}\|_2 \\ &= \|(-1)(\mathbf{u} - \mathbf{v})\|_2 \\ &= |-1|\|\mathbf{u} - \mathbf{v}\|_2 \\ &= \|\mathbf{u} - \mathbf{v}\|_2 = \text{dist}(\mathbf{u}, \mathbf{v})\end{aligned}$$

Summary

- Following this topic, you now
  - Know the definition of the 2-norm
  - Are aware of the properties of the 2-norm
  - Know the definition of the distance between two vectors

References

[1] https://en.wikipedia.org/wiki/Norm_(mathematics)

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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