

**UNIVERSITY OF WATERLOO ACULTY OF ENGINEERING** Department of Electrical & **Computer Engineering** 

NE 112 *Linear algebra for nanotechnology engineering*

## **6.4.9 Solutions to homogeneous systems of linear equations**



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# Introduction

- In this topic, we will
	- Review the definition of a homogenous system of linear equations
	- Compare solutions to non-homogenous and corresponding homogenous systems of linear equations
	- See that the zero vector is always a solution to a homogenous system of linear equations
	- Look at a number of results concerning all solutions to a homogenous system of linear equations





# Solving systems of linear equations

- Recall that a linear equation is *homogeneous* if the right-hand side or constant is zero
	- The term *homogeneous* implies all terms are of the same format
	- In a homogeneous linear equation, all terms are of the form a coefficient times an unknown
		- There is no constant term
- A system of homogeneous linear equations is equivalent to asking if a linear combination of vectors equals the zero vector

$$
\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}_m
$$





## Terminology

- Given a system of homogeneous linear equations, we will call it a *homogenous system of linear equations*
- Also, given a system of linear equations, the *corresponding homogeneous system* is that system of linear equations where all equations are equated to 0 Nutions to homogeneous systems of fine<br>
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	- For example, given

$$
3x1 + 5x2 - 4x3 = 2
$$
  
\n
$$
-x1 + 9x2 - 6x3 = 0
$$
  
\n
$$
7x1 - 2x2 - 8x3 = -1
$$
  
\n
$$
homog
$$

 $x_1 + 5x_2 - 4x_3 = 2$  being homogenous does  $x_1 + 9x_2 - 6x_3 = 0$  not make the system  $3x_1 + 5x_2 - 4x_3 = 2$  important. The equation  $7x, -2x, -8x, = -1$  homogenous Important: one equation

- The corresponding homogenous system is:
	-





### Terminology

- A non-homogeneous system of linear equations is any system where at least one linear equation is non-homogenous
	- This corresponds to the question of any linear combination of vectors equaling a non-zero target vector

$$
\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{v} \neq \mathbf{0}_m
$$

• If any one entry of the target vector is not zero, the corresponding system of linear equations is a non-homogenous system







# Solving systems of linear equations

- Recall that for a general system of linear equations, there may be:
	- No solutions
	- One unique solution
	- Infinitely many solutions





### At least one solution

- For a homogeneous system of linear equations, there is guaranteed to be at least one solution:
	- That solution is when all unknowns are 0, for given

$$
3x_1 + \cdots + 6x_n = 0
$$

has the solution

$$
3x_1 + \dots + 6x_n = 0
$$
  
\n
$$
x_1 = \dots = x_n = 0
$$
  
\n
$$
a_1 + \dots + a_n a_n = 0_m
$$

– Similarly, looking to solve

$$
\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}_m
$$
  
solution  

$$
\alpha_1 = \dots = \alpha_n = 0
$$

this also has the solution

$$
\alpha_1 = \cdots = \alpha_n = 0
$$







# Possibly infinitely many solutions

- There may be more than one solution to a homogeneous system of linear equations
	- For example,

$$
2x_1 + 3x_2 = 0
$$

has infinitely many solutions:

$$
\mathbf{x} = \begin{pmatrix} -1.5x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}
$$

 $-$  Note that when  $x_2 = 0$ , then  $x_1 = 0$ 

- We will call the solution when all unknowns are zero to be the *trivial solution*
	- The trivial solution is never a solution to a non-homogenous system of linear equations



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# Solutions for a homogenous system

- Thus, for a homogenous system of linear equations, there may be:
	- One unique solution; that is, the trivial solution
	- Infinitely many solutions, one of which is the trivial solution
- Recall that in Gaussian elimination, all we do is add multiples of one row onto another Solutions to homogeneous systems of theat equations<br>
utions for a homogenous systems of theat equations<br>
a homogenous system of linear equations,<br>  $\sum_{i=1}^{n}$  a homogenous system of linear equations,<br>  $\sum_{i=1}^{n}$  a homo
	- If one column (in this case, the last column) is all zeros, you can never get a non-zero entry appearing in that column
	- Thus, rank $(A)$  = rank $(A:0)$  for any matrix *A*, so it can never happen that there are no solutions





### Example

• Recall that previously, we found that this system of linear equations had one unique solution

$$
\begin{pmatrix}\n-3.2 & 2.6 & -2.4 & -25.4 \\
3.6 & -0.8 & 11 & 38.8 \\
4 & 3 & -2 & 3\n\end{pmatrix} \n\qquad\n\mathbf{x} = \begin{pmatrix}\n4 \\
-3 \\
2\n\end{pmatrix}
$$

• If you were to solve the corresponding homogeneous system of linear equations, you would get only the trivial solution:

3.2 2.6 2.4 0 3.6 0.8 11 0 4 3 2 0 − − <sup>−</sup> <sup>−</sup> 0 0 0 = **x**





## Example

• Previously, we found that this system of linear equations infinitely many solutions:

$$
\begin{pmatrix} 5 & 7 & -3 & | & 66 \\ 0 & 0 & -9 & | & 63 \\ 0 & 0 & 2.7 & | & -18.9 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 9 - 1.4x_2 \\ x_2 \\ -7 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ -7 \end{pmatrix} + x_2 \begin{pmatrix} -1.4 \\ 1 \\ 0 \end{pmatrix}
$$

• If you were to solve the corresponding homogeneous system of linear equations, the constant vector in the solution is zero:

$$
\begin{pmatrix} 5 & 7 & -3 & | & 0 \\ 0 & 0 & -9 & | & 0 \\ 0 & 0 & 2.7 & | & 0 \end{pmatrix} \qquad \qquad \mathbf{x} = \begin{pmatrix} -1.4x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1.4 \\ 1 \\ 0 \end{pmatrix}
$$





### Example

• Previously, we found that this system of linear equations had no solutions:

$$
\begin{pmatrix}\n-3 & 2 & 4 & 3 \\
1.8 & -2.7 & -2.1 & -2 \\
-0.6 & -4.6 & 1.8 & -3.4\n\end{pmatrix}
$$

• If you were to solve the corresponding homogeneous system of linear equations, you would find infinitely many solutions:

**Example**  
\n• Previously, we found that this system of linear equations had no solutions:  
\n
$$
\begin{pmatrix}\n-3 & 2 & 4 & 3 \\
1.8 & -2.7 & -2.1 & -2 \\
-0.6 & -4.6 & 1.8 & -3.4\n\end{pmatrix}
$$
\n• If you were to solve the corresponding homogeneous system of linear equations, you would find infinitely many solutions:  
\n
$$
\begin{pmatrix}\n-3 & 2 & 4 & 0 \\
1.8 & -2.7 & -2.1 & 0 \\
-0.6 & -4.6 & 1.8 & 0\n\end{pmatrix} \sim \begin{pmatrix}\n-3 & 2 & 4 & 0 \\
0 & -5 & 1 & 0 \\
0 & 0 & 0 & 0\n\end{pmatrix}
$$
\n
$$
\mathbf{x} = x_3 \begin{pmatrix}\n22 \\
3 \\
15\n\end{pmatrix}
$$





## Example

• Previously, we found that this system of linear equations infinitely many solutions:

$$
\begin{pmatrix} 0 & 2 & -5 & 9 \ 0 & 3 & 4.8 & -1.26 \ 0 & -5 & 2 & -9.9 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \ 1.5 \ -1.2 \end{pmatrix} = \begin{pmatrix} 0 \ 1.5 \ -1.2 \end{pmatrix} + x_1 \begin{pmatrix} 1 \ 0 \ 0 \end{pmatrix}
$$

• If you were to solve the corresponding homogeneous system of linear equations, the constant vector in the solution is zero:

$$
\begin{pmatrix} 0 & 2 & -5 & 0 \\ 0 & 3 & 4.8 & 0 \\ 0 & -5 & 2 & 0 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$





### Example

• Recall that previously, we found that this system of linear equations had one unique solution  $0.3$  )  $(0.3)$ 



• If you were to solve the corresponding homogeneous system of linear equations, you would get only the trivial solution:

$$
\begin{pmatrix}\n3.6 & 3.9 & -5.1 & 4.2 & 0 \\
-2.4 & -0.9 & 3.7 & 6.4 & 0 \\
4 & 3 & -1 & 0 & 0 \\
-2.8 & 0.9 & 2.7 & 3 & 0\n\end{pmatrix} \n\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$





### Example

• This system had infinitely many solutions:

**EXAMPLE 1** Solutions to homogeneous systems of the  
\n**Example**  
\n• This system had infinitely many solutions:  
\n
$$
\begin{pmatrix}\n2 & 4 & -1 & 3 & 5 & 11 \\
1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 16.6 \\
-0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 9.7\n\end{pmatrix}
$$
\n
$$
\mathbf{x} = \begin{pmatrix}\n6.8 \\
0 \\
2.6 \\
0 \\
0\n\end{pmatrix} + x_2 \begin{pmatrix}\n-2 \\
1 \\
0 \\
0 \\
0 \\
0\n\end{pmatrix} + x_4 \begin{pmatrix}\n-1.3 \\
0 \\
0.4 \\
1 \\
0\n\end{pmatrix} + x_5 \begin{pmatrix}\n-2.8 \\
-0.6 \\
0 \\
1\n\end{pmatrix}
$$
\n• Here is the solution for the corresponding homogeneous system:  
\n
$$
\begin{pmatrix}\n2 & 4 & -1 & 3 & 5 & 0 \\
1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 0 \\
-0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 0\n\end{pmatrix}
$$
\n
$$
\mathbf{x} = x_2 \begin{pmatrix}\n-2 \\
1 \\
0 \\
0 \\
0\n\end{pmatrix} + x_4 \begin{pmatrix}\n-1.3 \\
0 \\
0.4 \\
0\n\end{pmatrix} + x_5 \begin{pmatrix}\n-2.8 \\
0 \\
-0.6 \\
0\n\end{pmatrix}
$$
\n
$$
\mathbf{x} = x_2 \begin{pmatrix}\n-1.3 \\
0 \\
0 \\
0 \\
0\n\end{pmatrix} + x_4 \begin{pmatrix}\n-2.8 \\
0 \\
0.4 \\
0\n\end{pmatrix} + x_5 \begin{pmatrix}\n-2.8 \\
0 \\
-0.6 \\
0 \\
1\n\end{pmatrix}
$$
\n
$$
\mathbf{x} = x_2 \begin{pmatrix}\n-1.3 \\
0 \\
0 \\
0 \\
1\n\end{pmatrix} + x_5 \begin{pmatrix}\n-2.8 \\
0 \\
-0.6 \\
0 \\
1\n\end{pmatrix}
$$

• Here is the solution for the corresponding homogenous system:

$$
\begin{pmatrix} 2 & 4 & -1 & 3 & 5 & 0 \ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 0 \ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 0 \ \end{pmatrix}
$$

Equations:

\n
$$
= \begin{bmatrix}\n6.8 \\
0 \\
2.6 \\
-1 \\
0\n\end{bmatrix} + x_{2}\begin{bmatrix}\n-2 \\
1 \\
0 \\
0 \\
0\n\end{bmatrix} + x_{4}\begin{bmatrix}\n-1.3 \\
0 \\
0.4 \\
1\n\end{bmatrix} + x_{5}\begin{bmatrix}\n-2.8 \\
0 \\
-0.6 \\
0 \\
1\n\end{bmatrix}
$$
\nonding homogeneous system:

\n
$$
\mathbf{x} = x_{2}\begin{bmatrix}\n-2 \\
1 \\
0 \\
0 \\
0\n\end{bmatrix} + x_{4}\begin{bmatrix}\n-1.3 \\
0 \\
0.4 \\
1\n\end{bmatrix} + x_{5}\begin{bmatrix}\n-2.8 \\
0 \\
-0.6 \\
0 \\
1\n\end{bmatrix}
$$
\n
$$
\mathbf{x} = x_{2}\begin{bmatrix}\n-2 \\
1 \\
0 \\
0 \\
0\n\end{bmatrix} + x_{4}\begin{bmatrix}\n-1.3 \\
0 \\
0.4 \\
1\n\end{bmatrix} + x_{5}\begin{bmatrix}\n-2.8 \\
0 \\
-0.6 \\
0 \\
1\n\end{bmatrix}
$$





### Example

• Recall that previously, we found that this system of linear equations had one unique solution



• If you were to solve the corresponding homogeneous system of linear equations, you would get only the trivial solution:

$$
\begin{pmatrix}\n0 & 7.2 & -2.4 & 0 \\
1.5 & 0.9 & 10.3 & 0 \\
3.5 & 2.9 & 0.1 & 0 \\
5 & 3 & 1 & 0 \\
-2 & -9.2 & 5.6 & 0\n\end{pmatrix} \n\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$





# Relationships

- Thus, we may note, given a non-homogenous system:
	- If the corresponding homogeneous system of equations has only the trivial solution,

then the non-homogenous system either:

- Has a unique non-zero solution
- Has no solutions
- If the corresponding homogenous system of equations has infinitely many solutions,

then the non-homogenous system either:

- Also has infinitely many solutions (but **0** is not one of them)
- Has no solutions





### Theorems

### Theorem

If **x** is a solution to a system of linear equations, and  $\mathbf{x}_0$  is a solution to the corresponding homogenous system, then  $\mathbf{x} + \alpha \mathbf{x}_0$  is also a solution to the system of linear equations.

Proof: Let 
$$
\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{v}
$$
 be the system of linear equations.  
\nIf  $x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n = \mathbf{v}$  and  $x_{0,1} \mathbf{u}_1 + \cdots + x_{0,n} \mathbf{u}_n = \mathbf{0}_m$ ,  
\nthus  $(x_1 + \alpha x_{0,1}) \mathbf{u}_1 + \cdots + (x_n + \alpha x_{0,n}) \mathbf{u}_n$   
\n $= (x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n) + \alpha (x_{0,1} \mathbf{u}_1 + \cdots + x_{0,n} \mathbf{u}_n)$   
\n $= \mathbf{v} + \alpha \mathbf{0}_m = \mathbf{v} \blacksquare$ 







### Theorems

• For example,  $\mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix}$  is a solution to • Also,  $\mathbf{x}_0 = \begin{vmatrix} -2 \\ \end{vmatrix}$  is a solution to • Thus,  $\begin{vmatrix} 0 & +\alpha & -2 \end{vmatrix}$  is also a solution to 5 7 3 66  $0 \t -9 \t 63$  $0 \quad 0 \quad 2.7 = -18.9$  $\begin{pmatrix} 5 & 7 & -3 \end{pmatrix}$  66 ) −  $\begin{bmatrix} 9 \ 0 \ 7 \end{bmatrix}$  is a solution to  $\begin{bmatrix} 5 & 7 & -3 & | & 66 \ 0 & 0 & -9 & | & 63 \ 0 & 0 & 2.7 & | & -18.9 \end{bmatrix}$  $0<sup>1</sup>$  is a solution to  $10<sup>0</sup>$  -9 7 | (0 0 2.  $(9)$  (5 7  $= 0$  is a solution to  $\begin{bmatrix} 0 & 0 & -9 \ 0 & 0 & -9 \end{bmatrix}$  $(-7)$  (0 0 2.7  $\mathbf{X} = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$  is a solution to  $\left[ \begin{array}{c} 0 \\ 0 \end{array} \right]$  $5$  7  $-3$  0  $0\quad 0\quad -9\quad 0$  $0\quad \quad 0\quad \quad 2.7\pm 0$  $\begin{pmatrix} 5 & 7 & -3 & \vdots 0 \end{pmatrix}$  $\mathbf{p}_0 = \begin{bmatrix} -2 & \phantom{-} \ 0 & \phantom{-} \end{bmatrix}$  is a solution to  $\begin{bmatrix} 0 & \phantom{-}0 & -9 \ 0 & \phantom{-}0 & \phantom{-}2.7 \end{bmatrix}$   $\begin{bmatrix} 0 \phantom{-}0 \end{bmatrix}$  $(5 \t 7 \t -3$ 2 | is a solution to  $\begin{vmatrix} 0 & 0 & -9 \end{vmatrix}$  $\begin{array}{ccccccccc} 0 & \end{array}$   $\begin{array}{ccccccccc} 0 & 0 & 2 \end{array}$ Solutions to homogeneous systems of the are equations<br>
Theorems<br>  $\begin{pmatrix} 9 \\ -7 \end{pmatrix}$  is a solution to  $\begin{pmatrix} 5 & 7 & -3 & 66 \\ 0 & 0 & -9 & 63 \\ 0 & 0 & 2.7 & -18.9 \end{pmatrix}$ <br>  $\begin{pmatrix} 2.8 \\ -2 \\ 0 \end{pmatrix}$  is a solution to  $\begin{pmatrix} 5 & 7 & -3 & 0 \\$ Solutions to homogeneous systems of thesa equations<br>
Theorems<br>
mple,  $\mathbf{x} = \begin{pmatrix} 9 \\ 0 \\ -7 \end{pmatrix}$  is a solution to  $\begin{pmatrix} 5 & 7 & -3 & 66 \\ 0 & 0 & -9 & 63 \\ 0 & 0 & 2.7 & -18.9 \end{pmatrix}$ <br>  $= \begin{pmatrix} 2.8 \\ -2 \\ 0 \end{pmatrix}$  is a solution to  $\begin{pm$ Solutions to homogeneous systems of finear equations<br>
Theorems<br>  $\text{ple, } \mathbf{x} = \begin{pmatrix} 9 \\ 0 \\ -7 \end{pmatrix}$  is a solution to  $\begin{pmatrix} 5 & 7 & -3 & 66 \\ 0 & 0 & -9 & 63 \\ 0 & 0 & 2.7 & -18.9 \end{pmatrix}$ <br>  $\begin{pmatrix} 2.8 \\ -2 \\ 0 \end{pmatrix}$  is a solution to  $\begin{pm$  $\mathbf{x}_0 = \begin{vmatrix} -2 & \end{vmatrix}$  is a solution to  $\begin{vmatrix} 0 & \end{vmatrix}$  $9 \mid 2.8 \mid$  (5 7)  $0$   $\left| { + \alpha } \right|$   $- 2$   $\left| { \right|}$  is also a solution to  $\left| {\left. { \right|} \right\rangle}$  0  $\left| {\left. { \right|} \right\rangle}$ 7 ) (0)  $\alpha$   $-2$  - 1 is also a solution to 1 0 = 0  $(9)$   $(2.8)$  + −  $(-7)$  (0) 5 7 3 66  $0 \t -9 \t 63$  $0 \quad 0 \quad 2.7 \pm 18.9$  $(5 \t 7 \t -3 \t 66)$  $\begin{bmatrix} 0 & 0 & -9 \ 0 & 0 & 2.7 \end{bmatrix} \begin{bmatrix} 63 \ -18.9 \end{bmatrix}$ 







### Theorem

A system of linear equations has a unique solution only if the corresponding homogeneous system has only the trivial solution

Proof: Suppose a system of linear equations had two solutions: Consider  $\alpha - \beta$ But,  $(\alpha_1 - \beta_1) \mathbf{u}_1 + \cdots + (\alpha_n - \beta_n) \mathbf{u}_n$  $\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n = \mathbf{v}$ <br>  $\beta_1 \mathbf{u}_1 + \dots + \beta_n \mathbf{u}_n = \mathbf{v}$ <br>  $\alpha_1 - \beta_1 \mathbf{u}_1 + \dots + (\alpha_n - \beta_n) \mathbf{u}_n$ <br>  $= (\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n) - (\beta_1 \mathbf{u}_1 + \dots + \beta_n \mathbf{u}_n)$ <br>  $= \mathbf{v} - \mathbf{v} = \mathbf{0}_m$ <br>  $\alpha - \beta = \mathbf{0}_n$  and so  $\alpha$  $\boldsymbol{\beta}_n \mathbf{u}_n = \mathbf{v}$   $\boldsymbol{\beta}_1 \mathbf{u}_1 + \cdots + \boldsymbol{\beta}_n \mathbf{u}_n = \mathbf{v}$ =+  $\bullet$  +  $\bullet$  + ler  $\alpha - \beta$ <br>  $\alpha_1 - \beta_1 \mathbf{u}_1 + \cdots + (\alpha_n - \beta_n) \mathbf{u}_n$ <br>  $= (\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) - (\beta_1 \mathbf{u}_1 + \cdots + \beta_n)$ <br>  $= \mathbf{v} - \mathbf{v} = \mathbf{0}_m$ <br>  $\alpha - \beta = \mathbf{0}_n$  and so  $\alpha = \beta$ 

Thus,  $\boldsymbol{\alpha} - \boldsymbol{\beta} = \boldsymbol{0}_n$  and so  $\boldsymbol{\alpha} = \boldsymbol{\beta}$  **iii** 





### Theorems

### Theorem

The collection of all solutions to a homogeneous system of linear equations always forms a subspace of **F** *n* .

Proof: If 
$$
\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}_m
$$
 and  $\beta_1 \mathbf{u}_1 + \cdots + \beta_n \mathbf{u}_n = \mathbf{0}_m$   
\nThus,  $(\gamma \alpha_1) \mathbf{u}_1 + \cdots + (\gamma \alpha_n) \mathbf{u}_n = \gamma (\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) = \gamma \mathbf{0}_m = \mathbf{0}_m$   
\n $(\alpha_1 + \beta_1) \mathbf{u}_1 + \cdots + (\alpha_n + \beta_n) \mathbf{u}_n$   
\n $= (\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) + (\beta_1 \mathbf{u}_1 + \cdots + \beta_n \mathbf{u}_n)$   
\n $= \mathbf{0}_m + \mathbf{0}_m = \mathbf{0}_m$ 





### Theorems

Theorem

The collection of all solutions to a non-homogeneous system of linear equations is never a subspace of **F** *n* .

Proof: If the system is non-homogenous, then **0***<sup>n</sup>* is not a solution. If **0***<sup>n</sup>* is not in a collection of vectors, that collection of vectors cannot be a subspace. ▮





### **Observation**

- Because the solution to a homogenous system of linear equations is always a subspace of F *n* , we can always write the set of all solutions as the span of a set of vectors
- For example,

all solutions to  $-$  This equals  $-$  (  $2$  4  $-1$  3 5  $0$  $1.6$   $3.2$   $2.2$   $1.2$   $5.8 \pm 0$  $0.6$   $-1.2$   $5.3$   $-2.9$   $1.5 \pm 0$  $\left( \begin{array}{cccc} 2 & \phantom{0}4 & -1 & \phantom{0}3 & \phantom{0}5 & \phantom{0}0 \end{array} \right)$   $\begin{bmatrix} 1.6 & 3.2 & 2.2 & 1.2 & 5.8 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 \end{bmatrix}$   $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$   $\begin{bmatrix} x = x_2 \\ 0 \end{bmatrix}$   $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$   $\begin{bmatrix} +x_4 \\ 1 \end{bmatrix}$   $\begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$   $\begin{bmatrix} +x_5 \\ 1 \end{bmatrix}$ 2)  $(-1.3)$   $(-2.8)$ 1 0 0 0  $0 + x_1 \cdot 0.4 + x_5 \cdot -0.6$  $0$  | 1 | 1 | 0 0 1 1  $x_2$  | **U**  $x_3$  | **U**  $x_4$  |  $+x_5$  $(-2)$   $(-1.3)$   $(-2.8)$   $= x_2 \begin{vmatrix} 0 & +x_4 \end{vmatrix}$  0.4  $+ x_5 \begin{vmatrix} -0.6 \end{vmatrix}$   $\left(\begin{array}{c} 0 \end{array}\right)$   $\left(\begin{array}{c} 0 \end{array}\right)$   $\left(\begin{array}{c} 1 \end{array}\right)$ **x**  $2 \setminus (-1.3 \setminus (-2.8))$ 1 0 0  $\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} -1.3 \\ 0 \end{pmatrix} \begin{pmatrix} -2.8 \\ 0 \end{pmatrix} \right]$ 

$$
\text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \end{pmatrix}, \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \end{pmatrix} \right\}
$$





### **Observation**

- Suppose you have found one particular solution to a nonhomogeneous system of linear equations
- Suppose, also, that you know all solutions to the corresponding homogenous system
- If you add the particular solution to each solution of the corresponding homogenous system, you get all solutions to the non-homogenous system of linear equations





### **Observation**

• For example,

2 4  $-1$  3 5 0  $x=$  2  $1.6$   $3.2$   $2.2$   $1.2$   $5.8$   $6$   $1$   $10$   $1$ 0.6  $-1.2$  5.3  $-2.9$  1.5 10  $(0)$   $(-2)(-1.3)$  $\begin{pmatrix} 2 & 4 & -1 & 3 & 5 & 0 \end{pmatrix}$   $x = |2|$   $\left(\begin{matrix} -0.6 & -1.2 & 5.3 & -2.9 & 1.5 & \end{matrix} \begin{matrix} 10 \end{matrix}\right) \qquad \left(\begin{matrix} 0 \end{matrix}\right) \qquad \left|\left(\begin{matrix} -2 \end{matrix}\right)\right| \cdot$  $2 \left\lfloor \frac{2}{3} \right\rfloor$  $0<sup>1</sup>$ 

• Previously, we saw all solutions to the corresponding homogenous system were  $\lvert \text{span} \rvert \rvert = 0.4 \rvert , \rvert = 0.6$ 

$$
\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}
$$
  
the  
n were span 
$$
\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2.8 \\ 0 \\ -0.6 \\ 1 \\ 1 \end{bmatrix}
$$

 $1 \rangle$ 

 $(1)$ 

 $0<sup>1</sup>$ 

• Thus, all solutions to the non-homogenous system of linear equations are  $(1)$   $(-2)$   $(-1.3)$   $(-2.8)$ 

$$
\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix}
$$

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### Nice results

- It is useful to remember that:
	- $-$  If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are solutions to a homogenous system of linear equations, then so is  $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$
	- $-$  If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are solutions to a non-homogenous system of linear equations, then neither  $\mathbf{x}_1 + \mathbf{x}_2$  nor  $\alpha \mathbf{x}_1$  for  $\alpha \neq 1$  are solutions to that non-homogenous system
	- $-$  If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are solutions to a non-homogenous system of linear equations, then  $\mathbf{x}_1 - \mathbf{x}_2$  is a solution to the corresponding homogenous system
	- $-$  If  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are solutions to a non-homogenous system of linear equations, then  $\alpha$   $\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$  is also a solution to that nonhomogenous system for all scalar values of  $\alpha$



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## **Summary**

- Following this topic, you now
	- Understand what a homogenous system of linear equations is
	- Know that the zero vector is a solution to a homogenous system
		- All solutions to a homogenous system of linear equations is a subspace
		- All solutions to a non-homogenous system is never a subspace
	- Understand there is a strong relationship between solutions to a non-homogenous system of linear equations and the corresponding homogenous system





### References

### [1] https://en.wikipedia.org/wiki/System\_of\_linear\_equations #Homogeneous\_systems





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None so far.



 $\delta\,0\, \delta$ 



# Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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