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NE 112 *Linear algebra for nanotechnology engineering*

6.4.9 Solutions to homogeneous systems of linear equations

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Introduction

- In this topic, we will
  - Review the definition of a homogenous system of linear equations
  - Compare solutions to non-homogenous and corresponding homogenous systems of linear equations
  - See that the zero vector is always a solution to a homogenous system of linear equations
  - Look at a number of results concerning all solutions to a homogenous system of linear equations

Solving systems of linear equations

- Recall that a linear equation is *homogeneous* if the right-hand side or constant is zero
  - The term *homogeneous* implies all terms are of the same format
  - In a homogeneous linear equation,
    all terms are of the form a coefficient times an unknown
    - There is no constant term

- A system of homogeneous linear equations is equivalent to asking if a linear combination of vectors equals the zero vector

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}_m$$

Terminology

- Given a system of homogeneous linear equations,
  we will call it a *homogenous system of linear equations*
- Also, given a system of linear equations,
  the *corresponding homogeneous system* is that system of linear equations where all equations are equated to 0
  - For example, given

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 2 \\ -x_1 + 9x_2 - 6x_3 &= 0 \\ 7x_1 - 2x_2 - 8x_3 &= -1 \end{aligned}$$

  - The corresponding homogenous system is:

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -x_1 + 9x_2 - 6x_3 &= 0 \\ 7x_1 - 2x_2 - 8x_3 &= 0 \end{aligned}$$

Important: one equation being homogenous does not make the system homogenous

Terminology

- A non-homogeneous system of linear equations is any system where at least one linear equation is non-homogenous
  - This corresponds to the question of any linear combination of vectors equaling a non-zero target vector

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{v} \neq \mathbf{0}_m$$

- If any one entry of the target vector is not zero,
  the corresponding system of linear equations is a non-homogenous system

Solving systems of linear equations

- Recall that for a general system of linear equations, there may be:
  - No solutions
  - One unique solution
  - Infinitely many solutions

At least one solution

- For a homogeneous system of linear equations, there is guaranteed to be at least one solution:
  - That solution is when all unknowns are 0, for given

    $$3x_1 + \cdots + 6x_n = 0$$

    has the solution

    $$x_1 = \cdots = x_n = 0$$

  - Similarly, looking to solve

    $$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}_m$$

    this also has the solution

    $$\alpha_1 = \cdots = \alpha_n = 0$$

Possibly infinitely many solutions

- There may be more than one solution to a homogeneous system of linear equations
  - For example,

$$2x_1 + 3x_2 = 0$$

has infinitely many solutions:

$$\mathbf{x} = \begin{pmatrix} -1.5x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}$$

  - Note that when $x_2 = 0$, then $x_1 = 0$

- We will call the solution when all unknowns are zero to be the *trivial solution*
  - The trivial solution is never a solution to a non-homogenous system of linear equations

Solutions for a homogenous system

- Thus, for a homogenous system of linear equations, there may be:
  - One unique solution; that is, the trivial solution
  - Infinitely many solutions, one of which is the trivial solution

- Recall that in Gaussian elimination, all we do is add multiples of one row onto another
  - If one column (in this case, the last column) is all zeros, you can never get a non-zero entry appearing in that column
  - Thus, $\operatorname{rank}(A) = \operatorname{rank}(A \vdots \mathbf{0})$ for any matrix A, so it can never happen that there are no solutions

Example

- Recall that previously, we found that this system of linear equations had one unique solution

$$\left(\begin{array}{ccc:c} -3.2 & 2.6 & -2.4 & -25.4 \\ 3.6 & -0.8 & 11 & 38.8 \\ 4 & 3 & -2 & 3 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

- If you were to solve the corresponding homogeneous system of linear equations, you would get only the trivial solution:

$$\left(\begin{array}{ccc:c} -3.2 & 2.6 & -2.4 & 0 \\ 3.6 & -0.8 & 11 & 0 \\ 4 & 3 & -2 & 0 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

- Previously, we found that this system of linear equations infinitely many solutions:

$$\left(\begin{array}{ccc|c} 5 & 7 & -3 & 66 \\ 0 & 0 & -9 & 63 \\ 0 & 0 & 2.7 & -18.9 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 9 - 1.4x_2 \\ x_2 \\ -7 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ -7 \end{pmatrix} + x_2 \begin{pmatrix} -1.4 \\ 1 \\ 0 \end{pmatrix}$$

- If you were to solve the corresponding homogeneous system of linear equations, the constant vector in the solution is zero:

$$\left(\begin{array}{ccc|c} 5 & 7 & -3 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 2.7 & 0 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} -1.4x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1.4 \\ 1 \\ 0 \end{pmatrix}$$

Example

- Previously, we found that this system of linear equations had no solutions:

$$\left(\begin{array}{ccc:c} -3 & 2 & 4 & 3 \\ 1.8 & -2.7 & -2.1 & -2 \\ -0.6 & -4.6 & 1.8 & -3.4 \end{array}\right)$$

- If you were to solve the corresponding homogeneous system of linear equations, you would find infinitely many solutions:

$$\left(\begin{array}{ccc:c} -3 & 2 & 4 & 0 \\ 1.8 & -2.7 & -2.1 & 0 \\ -0.6 & -4.6 & 1.8 & 0 \end{array}\right) \sim \left(\begin{array}{ccc:c} -3 & 2 & 4 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathbf{x} = x_3 \begin{pmatrix} 22 \\ 3 \\ 15 \end{pmatrix}$$

Example

- Previously, we found that this system of linear equations infinitely many solutions:

$$\left(\begin{array}{ccc:c} 0 & 2 & -5 & 9 \\ 0 & 3 & 4.8 & -1.26 \\ 0 & -5 & 2 & -9.9 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ 1.5 \\ -1.2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1.5 \\ -1.2 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- If you were to solve the corresponding homogeneous system of linear equations, the constant vector in the solution is zero:

$$\left(\begin{array}{ccc:c} 0 & 2 & -5 & 0 \\ 0 & 3 & 4.8 & 0 \\ 0 & -5 & 2 & 0 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Example

- Recall that previously, we found that this system of linear equations had one unique solution

$$\left(\begin{array}{cccc|c} 3.6 & 3.9 & -5.1 & 4.2 & 13.14 \\ -2.4 & -0.9 & 3.7 & 6.4 & -4.34 \\ 4 & 3 & -1 & 0 & 4.8 \\ -2.8 & 0.9 & 2.7 & 3 & -3.06 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 0.3 \\ 0.7 \\ -1.5 \\ 0.4 \end{pmatrix}$$

- If you were to solve the corresponding homogeneous system of linear equations, you would get only the trivial solution:

$$\left(\begin{array}{cccc|c} 3.6 & 3.9 & -5.1 & 4.2 & 0 \\ -2.4 & -0.9 & 3.7 & 6.4 & 0 \\ 4 & 3 & -1 & 0 & 0 \\ -2.8 & 0.9 & 2.7 & 3 & 0 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

- This system had infinitely many solutions:

$$\left(\begin{array}{ccccc:c} 2 & 4 & -1 & 3 & 5 & 11 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 16.6 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 9.7 \end{array}\right)$$

$$\mathbf{x} = \begin{pmatrix} 6.8 \\ 0 \\ 2.6 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix}$$

- Here is the solution for the corresponding homogenous system:

$$\left(\begin{array}{ccccc:c} 2 & 4 & -1 & 3 & 5 & 0 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 0 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 0 \end{array}\right)$$

$$\mathbf{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix}$$

Example

- Recall that previously, we found that this system of linear equations had one unique solution

$$\left(\begin{array}{ccc|c} 0 & 7.2 & -2.4 & -15.84 \\ 1.5 & 0.9 & 10.3 & 17.67 \\ 3.5 & 2.9 & 0.1 & 3.97 \\ 5 & 3 & 1 & 8.9 \\ -2 & -9.2 & 5.6 & 19.04 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 2.5 \\ -1.7 \\ 1.5 \end{pmatrix}$$

- If you were to solve the corresponding homogeneous system of linear equations, you would get only the trivial solution:

$$\left(\begin{array}{ccc|c} 0 & 7.2 & -2.4 & 0 \\ 1.5 & 0.9 & 10.3 & 0 \\ 3.5 & 2.9 & 0.1 & 0 \\ 5 & 3 & 1 & 0 \\ -2 & -9.2 & 5.6 & 0 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Relationships

- Thus, we may note, given a non-homogenous system:
  - If the corresponding homogeneous system of equations has only the trivial solution,

    then the non-homogenous system either:
    - Has a unique non-zero solution
    - Has no solutions
  - If the corresponding homogenous system of equations has infinitely many solutions,

    then the non-homogenous system either:
    - Also has infinitely many solutions (but **0** is not one of them)
    - Has no solutions

Theorems

Theorem

If $\mathbf{x}$ is a solution to a system of linear equations, and
$\mathbf{x}_0$ is a solution to the corresponding homogenous system,
then $\mathbf{x} + \alpha\,\mathbf{x}_0$ is also a solution to the system of linear equations.

Proof: Let $\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{v}$ be the system of linear equations.

If $x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n = \mathbf{v}$ and $x_{0,1}\mathbf{u}_1 + \cdots + x_{0,n}\mathbf{u}_n = \mathbf{0}_m$,

thus

$$\begin{aligned}(x_1 + \alpha x_{0,1})\mathbf{u}_1 + \cdots + (x_n + \alpha x_{0,n})\mathbf{u}_n &= (x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n) + \alpha(x_{0,1}\mathbf{u}_1 + \cdots + x_{0,n}\mathbf{u}_n) \\ &= \mathbf{v} + \alpha\mathbf{0}_m = \mathbf{v} \quad \blacksquare\end{aligned}$$

Theorems

- For example, $\mathbf{x} = \begin{pmatrix} 9 \\ 0 \\ -7 \end{pmatrix}$ is a solution to $\left(\begin{array}{ccc:c} 5 & 7 & -3 & 66 \\ 0 & 0 & -9 & 63 \\ 0 & 0 & 2.7 & -18.9 \end{array}\right)$
- Also, $\mathbf{x}_0 = \begin{pmatrix} 2.8 \\ -2 \\ 0 \end{pmatrix}$ is a solution to $\left(\begin{array}{ccc:c} 5 & 7 & -3 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 2.7 & 0 \end{array}\right)$
- Thus, $\begin{pmatrix} 9 \\ 0 \\ -7 \end{pmatrix} + \alpha \begin{pmatrix} 2.8 \\ -2 \\ 0 \end{pmatrix}$ is also a solution to $\left(\begin{array}{ccc:c} 5 & 7 & -3 & 66 \\ 0 & 0 & -9 & 63 \\ 0 & 0 & 2.7 & -18.9 \end{array}\right)$

Theorems

Theorem

A system of linear equations has a unique solution only if the corresponding homogeneous system has only the trivial solution

Proof: Suppose a system of linear equations had two solutions:

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{v} \qquad \beta_1\mathbf{u}_1 + \cdots + \beta_n\mathbf{u}_n = \mathbf{v}$$

Consider $\boldsymbol{\alpha} - \boldsymbol{\beta}$

But, $(\alpha_1 - \beta_1)\mathbf{u}_1 + \cdots + (\alpha_n - \beta_n)\mathbf{u}_n$

$$= (\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n) - (\beta_1\mathbf{u}_1 + \cdots + \beta_n\mathbf{u}_n)$$
$$= \mathbf{v} - \mathbf{v} = \mathbf{0}_m$$

Thus, $\boldsymbol{\alpha} - \boldsymbol{\beta} = \mathbf{0}_n$ and so $\boldsymbol{\alpha} = \boldsymbol{\beta}$ ∎

Theorems

Theorem

The collection of all solutions to a homogeneous system of linear equations always forms a subspace of $\mathbf{F}^n$.

Proof: If $\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}_m$ and $\beta_1\mathbf{u}_1 + \cdots + \beta_n\mathbf{u}_n = \mathbf{0}_m$

Thus, $(\gamma\alpha_1)\mathbf{u}_1 + \cdots + (\gamma\alpha_n)\mathbf{u}_n = \gamma(\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n) = \gamma\mathbf{0}_m = \mathbf{0}_m$

$$\begin{aligned}(\alpha_1 + \beta_1)\mathbf{u}_1 + \cdots + (\alpha_n + \beta_n)\mathbf{u}_n \\ &= (\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n) + (\beta_1\mathbf{u}_1 + \cdots + \beta_n\mathbf{u}_n) \\ &= \mathbf{0}_m + \mathbf{0}_m = \mathbf{0}_m \quad \blacksquare\end{aligned}$$

Theorems

Theorem

The collection of all solutions to a non-homogeneous system of linear equations is never a subspace of $\mathbf{F}^n$.

Proof: If the system is non-homogenous, then $\mathbf{0}_n$ is not a solution.

If $\mathbf{0}_n$ is not in a collection of vectors, that collection of vectors cannot be a subspace. ∎

Observation

- Because the solution to a homogenous system of linear equations is always a subspace of F^n, we can always write the set of all solutions as the span of a set of vectors
- For example,

  all solutions to

$$\left(\begin{array}{ccccc|c} 2 & 4 & -1 & 3 & 5 & 0 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 0 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 0 \end{array}\right)$$

$$\mathbf{x} = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix}$$

  – This equals

$$\operatorname{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Observation

- Suppose you have found one particular solution to a non-homogeneous system of linear equations
- Suppose, also, that you know all solutions to the corresponding homogenous system
- If you add the particular solution to each solution of the corresponding homogenous system, you get all solutions to the non-homogenous system of linear equations

Observation

- For example,

$$\left(\begin{array}{ccccc:c} 2 & 4 & -1 & 3 & 5 & 0 \\ 1.6 & 3.2 & 2.2 & 1.2 & 5.8 & 6 \\ -0.6 & -1.2 & 5.3 & -2.9 & 1.5 & 10 \end{array}\right) \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

- Previously, we saw all solutions to the corresponding homogenous system were $\operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix} \right\}$

- Thus, all solutions to the non-homogenous system of linear equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1.3 \\ 0 \\ 0.4 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2.8 \\ 0 \\ -0.6 \\ 0 \\ 1 \end{pmatrix}$$

Nice results

- It is useful to remember that:
  - If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to a homogenous system of linear equations, then so is $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$
  - If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to a non-homogenous system of linear equations, then neither $\mathbf{x}_1 + \mathbf{x}_2$ nor $\alpha\,\mathbf{x}_1$ for $\alpha \neq 1$ are solutions to that non-homogenous system
  - If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to a non-homogenous system of linear equations, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to the corresponding homogenous system
  - If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to a non-homogenous system of linear equations, then $\alpha\,\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$ is also a solution to that non-homogenous system for all scalar values of α

Summary

- Following this topic, you now
  - Understand what a homogenous system of linear equations is
  - Know that the zero vector is a solution to a homogenous system
    - All solutions to a homogenous system of linear equations is a subspace
    - All solutions to a non-homogenous system is never a subspace
  - Understand there is a strong relationship between solutions to a non-homogenous system of linear equations and the corresponding homogenous system

References

[1] https://en.wikipedia.org/wiki/System_of_linear_equations#Homogeneous_systems

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

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