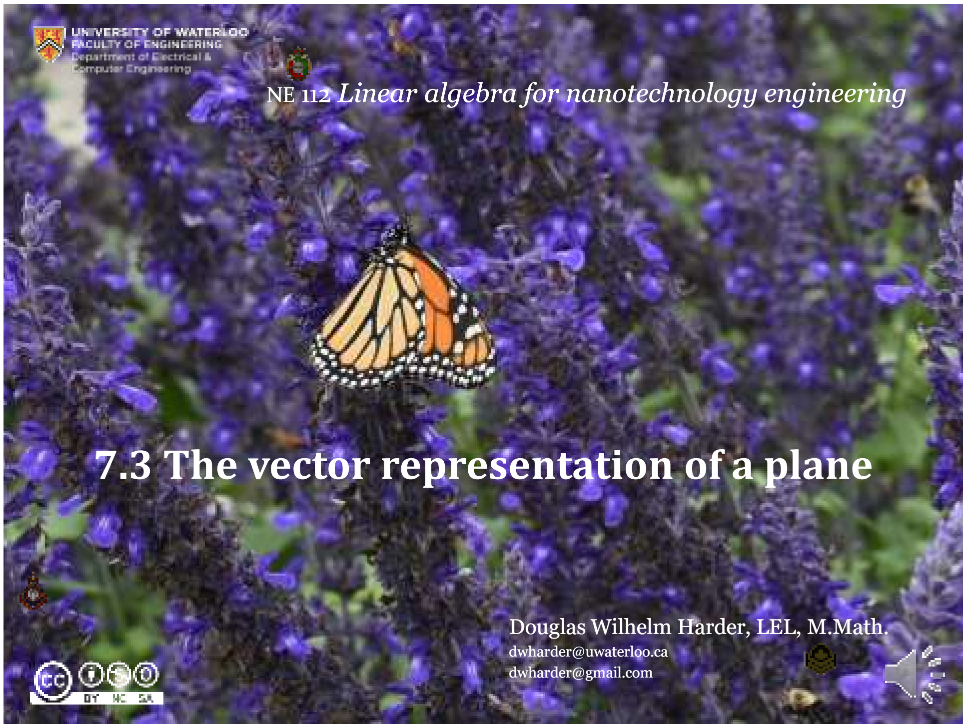

1

The vector representation of a plane

Introduction

- In this topic, we will
  - Describe the vector representation of a plane
  - Derive an ideal representation
  - Determine the point on the plane closest to a given point **w** using this ideal representation
  - Find a formula for the minimum distance from **w** to that plane

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Vector representation of a plane

- In $\mathbf{R}^3$,
  a vector representation of a plane is $\mathbf{u} + \alpha\,\mathbf{v}_1 + \beta\,\mathbf{v}_2$ where
  - $\mathbf{u}$ is any point on the plane
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent vectors in the plane
  - α and β are any real numbers
- As with lines, there are infinitely many representations of the same plane
  - For example, both of these describe the same plane:

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 926 \\ -311.5 \\ -640 \end{pmatrix} + \alpha \begin{pmatrix} 13.9 \\ 0.2 \\ 17.3 \end{pmatrix} + \beta \begin{pmatrix} -30 \\ 20.1 \\ -5.4 \end{pmatrix}$$

  - Question: again, is any one representation better than others?

Vector representation of a plane

- As with lines,
  your first guess may be to normalize both $\mathbf{v}_1$ and $\mathbf{v}_2$
  - Issue: $\mathbf{v}_1$ and $\mathbf{v}_2$ may not be perpendicular
  - Instead, apply the Gram-Schmidt algorithm to $\mathbf{v}_1$ and $\mathbf{v}_2$
  - Next, make $\mathbf{u}$ perpendicular to the plane
    - This will be the point on the plane that is closest to the origin, and therefore with the smallest 2-norm

Vector representation of a plane

- Thus, given $\mathbf{u} + \alpha\,\mathbf{v}_1 + \beta\,\mathbf{v}_2$, we proceed as follows:
  - Apply Gram-Schmidt, so the expression of our plane is now

$$\mathbf{u} + \alpha\hat{\mathbf{v}}_1 + \beta\hat{\mathbf{v}}_2$$

  - Second, find the component of $\mathbf{u}$ perpendicular both these vectors

$$\mathbf{u} \leftarrow \mathbf{u} - \langle \hat{\mathbf{v}}_1, \mathbf{u} \rangle \hat{\mathbf{v}}_1$$

$$\mathbf{u}_\perp \leftarrow \mathbf{u} - \langle \hat{\mathbf{v}}_2, \mathbf{u} \rangle \hat{\mathbf{v}}_2$$

  - Now, if $\mathbf{u}_\perp + \alpha\hat{\mathbf{v}}_1 + \beta\hat{\mathbf{v}}_2$ is a point on the plane, then the distance to the point $\mathbf{u}_\perp$ is

$$\sqrt{\alpha^2 + \beta^2}$$

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The vector representation of a plane

Vector representation of a plane

- For example, consider the plane described by:

$$\begin{pmatrix} 12 \\ -17 \\ -5 \end{pmatrix} + \alpha \begin{pmatrix} 16 \\ -12 \\ -15 \end{pmatrix} + \beta \begin{pmatrix} -68 \\ 51 \\ -30 \end{pmatrix}$$

  - First, normalize the vector $\mathbf{v}_1$:

$$\hat{\mathbf{v}}_1 \leftarrow \begin{pmatrix} 0.64 \\ -0.48 \\ -0.6 \end{pmatrix}$$

  - Next, subtract off the projection of $\mathbf{v}_2$ onto $\hat{\mathbf{v}}_1$ and then normalize

$$\mathbf{v}_2 \leftarrow \mathbf{v}_2 - \langle \hat{\mathbf{v}}_1, \mathbf{v}_2 \rangle \hat{\mathbf{v}}_1 = \begin{pmatrix} -36 \\ 27 \\ -60 \end{pmatrix} \qquad \hat{\mathbf{v}}_2 \leftarrow \begin{pmatrix} -0.48 \\ 0.36 \\ -0.8 \end{pmatrix}$$

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Vector representation of a plane

- We now have a representation of the plane:

$$\begin{pmatrix} 12 \\ -17 \\ -5 \end{pmatrix} + \alpha \begin{pmatrix} 0.64 \\ -0.48 \\ -0.6 \end{pmatrix} + \beta \begin{pmatrix} -0.48 \\ 0.36 \\ -0.8 \end{pmatrix}$$

  – However, **u** is not perpendicular to that plane
    - We must subtract from **u** the projection onto both $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$

$$\mathbf{u} \leftarrow \mathbf{u} - \langle \hat{\mathbf{v}}_1, \mathbf{u} \rangle \hat{\mathbf{v}}_1 = \begin{pmatrix} -0.0576 \\ -7.9568 \\ 6.304 \end{pmatrix}$$

$$\mathbf{u}_\perp \leftarrow \mathbf{u} - \langle \hat{\mathbf{v}}_2, \mathbf{u} \rangle \hat{\mathbf{v}}_2 = \begin{pmatrix} -3.84 \\ -5.12 \\ 0 \end{pmatrix}$$

Vector representation of a plane

- Thus, an ideal representation of the plane is:

$$\begin{pmatrix} -3.84 \\ -5.12 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0.64 \\ -0.48 \\ -0.6 \end{pmatrix} + \beta \begin{pmatrix} -0.48 \\ 0.36 \\ -0.8 \end{pmatrix}$$

- The shortest distance to the plane from the origin is

$$\|\mathbf{u}_\perp\|_2 = 6.4$$

Point on the plane closest to a given point

- Now, given we have $\mathbf{u}_\perp + \alpha\hat{\mathbf{v}}_1 + \beta\hat{\mathbf{v}}_2$,
  what is the point on this plane closest to a vector **w**?
  - Because $\mathbf{u}_\perp$ is already perpendicular to the plane, all we need to is project **w** onto the two basis vectors:
  $$\mathbf{u}_\perp + \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle \hat{\mathbf{v}}_1 + \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle \hat{\mathbf{v}}_2$$
- This is a more difficult problem for an arbitrary vector representation $\mathbf{u} + \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$

Point on the plane closest to a given point

- For example, what is the point on the plane
$$\begin{pmatrix} -3.84 \\ -5.12 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0.64 \\ -0.48 \\ -0.6 \end{pmatrix} + \beta \begin{pmatrix} -0.48 \\ 0.36 \\ -0.8 \end{pmatrix}$$
  that is closest to this point?
$$\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
  - Because $\mathbf{u}_\perp$ is orthogonal, we only need project **w**:
$$\mathbf{u}_\perp + \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle \hat{\mathbf{v}}_1 + \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle \hat{\mathbf{v}}_2 = \begin{pmatrix} -3.68 \\ -5.24 \\ 1 \end{pmatrix}$$

Point on the plane closest to a given point

- Given this closest point

$$\mathbf{u}_\perp + \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle \hat{\mathbf{v}}_1 + \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle \hat{\mathbf{v}}_2 = \begin{pmatrix} -3.68 \\ -5.24 \\ 1 \end{pmatrix}$$

we also have that the coefficients are

$$\alpha = \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle = -0.44$$
$$\beta = \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle = -0.92$$

- Thus, if we are using meters,
  this closest point is approximately 1.020 m from the point $\mathbf{u}_\perp$
  - You can verify yourself that this is indeed the closest point to $\mathbf{w}$

Minimum distance to a plane from a given $\mathbf{w}$

- You can also calculate the distance to the plane as follows:

$$\sqrt{\|\mathbf{u}_\perp\|_2^2 + \|\mathbf{w}\|_2^2 - 2\langle \mathbf{u}, \mathbf{w} \rangle - \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle^2 - \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle^2}$$

  - You can derive this yourself by calculating

$$\|\mathbf{u}_\perp + \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle \hat{\mathbf{v}}_1 + \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle \hat{\mathbf{v}}_2 - \mathbf{w}\|_2^2$$
$$= \langle \mathbf{u}_\perp + \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle \hat{\mathbf{v}}_1 + \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle \hat{\mathbf{v}}_2 - \mathbf{w}, \mathbf{u}_\perp + \langle \hat{\mathbf{v}}_1, \mathbf{w} \rangle \hat{\mathbf{v}}_1 + \langle \hat{\mathbf{v}}_2, \mathbf{w} \rangle \hat{\mathbf{v}}_2 - \mathbf{w} \rangle$$

  - In this example, the distance is 7.8 m

Summary

- Following this topic, you now
  - Understand the vector representation of a plane
  - Know how to find an ideal representation
  - Can use this representation to find the point on the plane closest to a given point **w**
  - Are aware of a formula to calculate the minimum distance from **w** to that plane

References

[1] https://en.wikipedia.org/wiki/Plane_(geometry)

Acknowledgments

None so far.

Colophon

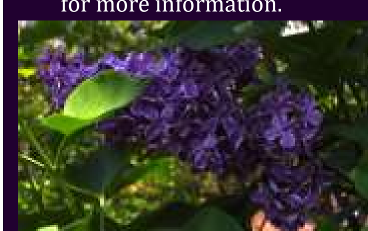
These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

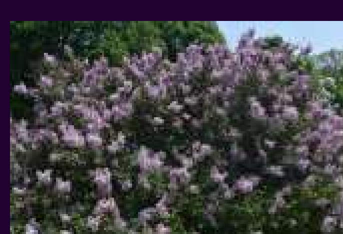
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.





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