Inductors and Capacitors

- **Inductor** is a coil of wire wrapped around a supporting (mag or non mag) core
- Inductor behavior related to magnetic field
- Current (movement of charge) is source of the magnetic field
- Time varying current sets up a time varying magnetic field
- Time varying magnetic field induces a voltage in any conductor linked by the field
- Inductance relates the induced voltage to the current

- **Capacitor** is two conductors separated by a dielectric insulator
- Capacitor behavior related to electric field
- Separation of charge (or voltage) is the source of the electric field
- Time varying voltage sets up a time varying electric field
- Time varying electric field generates a displacement current in the space of field
- Capacitance relates the displacement current to the voltage
- Displacement current is equal to the conduction current at the terminals of capacitor
Inductors and Capacitors (contd)

• Both inductors and capacitors can store energy (since both magnetic fields and electric fields can store energy)
• Ex, energy stored in an inductor is released to fire a spark plug
• Ex, Energy stored in a capacitor is released to fire a flash bulb
• L and C are passive elements since they do not generate energy
Inductor

- Inductance symbol $L$ and measured in Henrys (H)
- Coil is a reminder that inductance is due to conductor linking a magnetic field

\[ v = L \frac{di}{dt} \]

- First, if current is constant, $v = 0$
- Thus **inductor behaves as a short with dc current**
- Next, **current cannot change instantaneously in $L$** i.e. current cannot change by a finite amount in 0 time since an infinite (i.e. impossible) voltage is required
- In practice, when a switch on an inductive circuit is opened, current will continue to flow in air across the switch (arching)
Inductor: Voltage behavior

- Why does the inductor voltage change sign even though the current is positive? (slope)
- Can the voltage across an inductor change instantaneously? (yes)
Inductor: Current, power and energy

\[ v = L \frac{di}{dt} \]
\[ vdt = L \left( \frac{di}{dt} \right) dt \]
\[ vdt = Ldi \]
\[ Ldi = vdt \]
\[ i(t) \]
\[ L \int_{i(t_0)}^{i} dx = \int_{i(t_0)}^{i} vdt \]
\[ i(t) = \frac{1}{L} \int_{i(t_0)}^{i} vdt + i(t_0) \]
\[ t_0 = 0; i(t) = \frac{1}{L} \int_{0}^{t} vdt + i(0) \]

\[ p = vi \]
\[ p = \left( L \frac{di}{dt} \right) i \]
\[ p = v \left[ \frac{1}{L} \int_{i(t_0)}^{i} vdt + i(t_0) \right] \]
\[ p = \frac{dw}{dt} = Li \frac{di}{dt} \]
\[ \therefore dw = (Li)di \]
\[ \int_{0}^{i} dx = L \int_{0}^{i} ydy \]
\[ \therefore w = \frac{1}{2} Li^2 \]
• Why does the current approach a constant value (2A here) even though the voltage across the L is being reduced? (lossless element)
In this example, the excitation comes from a current source.

Initially increasing current up to 0.2s is storing energy in the inductor, decreasing current after 0.2 s is extracting energy from the inductor.

Note the positive and negative areas under the power curve are equal. When power is positive, energy is stored in L. When power is negative, energy is extracted from L.
Inductor: Example 6.3, V source

- In this example, the excitation comes from a voltage source.
- Application of positive voltage pulse stores energy in inductor.
- Ideal inductor cannot dissipate energy – thus a sustained current is left in the circuit even after the voltage goes to zero (lossless inductor).
- In this case energy is never extracted.
Capacitor

• Capacitance symbol C and measured in Farads (F)

• Air gap in symbol is a reminder that capacitance occurs whenever conductors are separated by a dielectric

• Although putting a V across a capacitor cannot move electric charge through the dielectric, it can displace a charge within the dielectric $\rightarrow$ displacement current proportional to $v(t)$

• At the terminals, displacement current is similar to conduction current

$$i = C \frac{dv}{dt}$$

• As per above eqn, **voltage cannot change instantaneously across the terminals of a capacitor** i.e. voltage cannot change by a finite amount in 0 time since an infinite (i.e. impossible) current would be produced

• Next, for DC voltage, capacitor current is 0 since conduction cannot happen through a dielectric (need a time varying voltage $v(t)$ to create a displacement current). Thus, **a capacitor is open circuit for DC voltages.**
Capacitor: voltage, power and energy

\[ i = C \frac{dv}{dt} \]

\[ idt = C \left( \frac{dv}{dt} \right) dt \]

\[ idt = Cdv \]

\[ dv = \frac{1}{C} idt \]

\[ \int_{v(t_0)}^{v(t)} dx = \frac{1}{C} \int_{t_0}^{t} id\tau \]

\[ \therefore v(t) = \frac{1}{C} \int_{t_0}^{t} id\tau + v(t_0) \]

\[ t_0 = 0; v(t) = \frac{1}{C} \int_{0}^{t} id\tau + v(0) \]

\[ p = vi \]

\[ p = v \left( \frac{C dv}{dt} \right) \]

\[ p = i \left[ \frac{1}{C} \int_{t_0}^{t} id\tau + v(t_0) \right] \]

\[ p = \frac{dw}{dt} \]

\[ \therefore dw = (Cv)dv \]

\[ \int_{0}^{v} dx = \int_{0}^{v} y dy \]

\[ \therefore w = \frac{1}{2} Cv^2 \]
In this example, the excitation comes from a voltage source.

Energy is being stored in the capacitor whenever the power is positive and delivered when the power is negative.

Voltage applied to capacitor returns to zero with increasing time. Thus, energy stored initially (up to 1 s) is returned over time as well.
Capacitor: Example 6.5, 1 source

- In this example, the excitation comes from a current source.

- Energy is being stored in the capacitor whenever the power is positive.

- Here since power is always positive, energy is continually stored in capacitor. When current returns to zero, the stored energy is trapped since ideal capacitor. Thus a voltage remains on the capacitor permanently (ideal lossless capacitor).

- Concept used extensively in memory and imaging circuits.
Series-Parallel Combination (L)

\[ i_1 = \frac{1}{L_1} \int_{t_0}^{t} v \, dt + i_1(t_0) \]
\[ i_2 = \frac{1}{L_2} \int_{t_0}^{t} v \, dt + i_2(t_0) \]
\[ i_3 = \frac{1}{L_3} \int_{t_0}^{t} v \, dt + i_3(t_0) \]
\[ i = i_1 + i_2 + i_3 \]
\[ i = \left( \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \int_{t_0}^{t} v \, dt + i_1(t_0) + i_2(t_0) + i_3(t_0) \]
\[ i = \left( \frac{1}{L_{eq}} \right) \int_{t_0}^{t} v \, dt + i(t_0) \]

\[ \text{Here,} \quad \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} ; i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0) \]
Series Combination (C)

\[ v = \frac{1}{C_1} \int_0^t i \, dt + v_1(t_0) + \frac{1}{C_2} \int_0^t i \, dx + v_2(t_0) + \cdots \]

\[ v = \left[ \frac{1}{C_1} + \frac{1}{C_2} + \cdots \right] \int_0^t i \, dx + v_1(t_0) + v_2(t_0) + \cdots \]

(a) \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}

\[ v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0) \]

(b)
Parallel Combination (C)

\[ i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \cdots = [C_1 + C_2 + \cdots] \frac{dv}{dt} \]

Therefore \( C_{eq} = C_1 + C_2 + \cdots \). Because the capacitors are in parallel, the initial voltage on every capacitor must be the same. This initial voltage would appear on \( C_{eq} \).

\[ C_{eq} = C_1 + C_2 + \cdots + C_n \]
First Order RL and RC circuits

- Class of circuits that are analyzed using first order ordinary differential equations.
- To determine circuit behavior when energy is released or acquired by L and C due to an abrupt change in dc voltage or current.
- **Natural response**: $i(t)$ and $v(t)$ when energy is released into a resistive network (i.e. when L or C is disconnected from its DC source).
- **Step response**: $i(t)$ and $v(t)$ when energy is acquired by L or C (due to the sudden application of a DC i or v)

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**Figure: 07-01**

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**Figure: 07-02**

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Natural response: RL circuit

- Assume all currents and voltages in circuit have reached steady state (constant, dc) values

Prior to switch opening,
- L is acting as short circuit (i.e. since at DC)
- So all $I_s$ is in L and none in R
- We want to find $v(t)$ and $i(t)$ for $t>0$

- Since current cannot change instantly in L, $i(0^-) = i(0^+) = I_0$

\[
L \frac{di}{dt} + Ri = 0
\]
\[
\frac{di}{dt} = -\frac{R}{L} idt
\]
\[
\frac{di}{i} = -\frac{R}{L} dt
\]
\[
\int_{i(0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L} \int_{t_0}^{t} dy; t_0 = 0
\]
\[
\ln \left( \frac{i(t)}{i(0)} \right) = -\frac{R}{L} t
\]
\[
\therefore i(t) = i(0)e^{-\left(\frac{R}{L}\right) t}
\]

- $v(0^-) = 0$ but $v(0^+) = I_0R$

\[
v = i(t)R = R I_0 e^{-\left(\frac{R}{L}\right) t}; t \geq 0^+
\]
\[
p = vi = i^2 R = \frac{v^2}{R} = R I_0^2 e^{-2\left(\frac{R}{L}\right) t}; t \geq 0^+
\]
\[
w = \int_{0}^{t} p dx = \frac{1}{2} LI_0^2 \left( 1 - e^{-2\left(\frac{R}{L}\right) t} \right); t \geq 0
\]
Natural response time constant

- Both i(t) and v(t) have a term
- Time constant τ is defined as

\[ \tau = \left( \frac{L}{R} \right) \]

\[ i(t) = I_0 e^{-\frac{R}{L}t}; t \geq 0 \]

\[ v(t) = RI_0 e^{-\frac{R}{L}t}; t \geq 0 \]

\[ p = RI_0^2 e^{-\frac{2R}{L}t}; t \geq 0 \]

\[ w = \frac{1}{2} LI_0^2 \left( 1 - e^{-\frac{2R}{L}t} \right); t \geq 0 \]

- Think of τ as an integral parameter
- i.e. after 1 τ, the inductor current has been reduced to e^{-1} (or 0.37) of its initial value. After 5 τ, the current is less than 1% of its original value (i.e. steady state is achieved)
- The existence of current in the RL circuit is momentary – transient response. After 5τ, cct has steady state response
Extracting \( \tau \)

- If \( R \) and \( L \) are unknown
- \( \tau \) can be determined from a plot of the natural response of the circuit
- For example,

\[
\frac{di}{dt}(0^+) = -\frac{1}{\tau}I_0e^{-\frac{v}{\tau}} = -\frac{R}{L}I_0 = -\frac{I_0}{\tau}
\]

- If \( i \) starts at \( I_0 \) and decreases at \( I_0/\tau \), \( i \) becomes

\[
\therefore i = I_0 - \frac{I_0}{\tau}t
\]

- Then, drawing a tangent at \( t = 0 \) would yield \( \tau \) at the x-axis intercept
- And if \( I_0 \) is known, natural response can be written as,

\[
i(t) = I_0e^{-\frac{t}{\tau}}
\]
Example 7.1

• To find $i_L(t)$ for $t \geq 0$, note that since cct is in steady state before switch is opened, $L$ is a short and all current is in it, i.e. $I_L(0^+) = I_L(0^-) = 20$A
• Simplify resistors with $R_{eq} = 2 + 40 \parallel 10 = 10\Omega$
• Then $\tau = \frac{L}{R} = 0.2$ s,
• With switch open,

$\therefore i_L(t) = 20e^{-\frac{t}{0.2}} A, t \geq 0$

$i_0 = -i_L\left(\frac{10}{10 + 40}\right) = -0.2i_L$

$\therefore i_0(t) = -4e^{-\frac{t}{0.2}} A, t \geq 0^+$

$v_0(t) = 40i_0(t) = -160e^{-\frac{t}{0.2}} V, t \geq 0^+$

$P_{10\Omega}(t) = \frac{v_0(t)^2}{10} = 2560e^{-\frac{t}{0.1}} W, t \geq 0^+$

$W_{10\Omega}(t) = \int_0^\infty 2560e^{-\frac{t}{0.1}} dt = 256J$

$w(0) = \frac{1}{2} Li^2(0) = 400J$

$\%\frac{W_{10\Omega}(t)}{w(0)} = 64\%$
Example 7.2

• Initial I in \( L_1 \) and \( L_2 \) already established by “hidden sources”

• To get \( i_1, i_2 \) and \( i_3 \), find \( v(t) \) (since parallel cct) with simplified circuit

\[
L = 4H, \quad R = 8\Omega, \\
\therefore i(t) = 12e^{-2t}A, \quad t \geq 0 \\
v_0(t) = 8i_0(t) = 96e^{-2t}V, \quad t \geq 0^+ \\
v_0(t) = 0, \quad t < 0
\]

\[
\therefore i(t) = \frac{1}{L} \int_0^t v(t) \, d\tau + i(t_0) \\
i_1(t) = \frac{1}{L} \int_0^t 96e^{-2x} \, dx = 8 \\
i_1(t) = 1.6 - 9.6e^{-2t}A, \quad t \geq 0 \\
i_2(t) = -1.6 - 2.4e^{-2t}A, \quad t \geq 0 \\
i_3(t) = \frac{v(t)}{R} = \frac{15}{10} = 5.76e^{-2t}A, \quad t \geq 0^+
\]

• Note inductor current \( i_1 \) and \( i_2 \) are valid from \( t \geq 0 \) since current in inductor cannot change instantaneously

• However, resistor current \( i_3 \) is valid only from \( t \geq 0^+ \) since there is 0 current in resistor at \( t = 0 \) (all I is shorted through inductors in steady state)
Example 7.2 (contd)

- Initial energy stored in inductors

\[ w = \frac{1}{2} Li^2; w_{init} = w_{5H} + w_{20H} \]

\[ w_{init} = \frac{1}{2} (5)(64) + \frac{1}{2} (20)(16) = 320J \]

\[ I_f(t \to \infty, i_1 \to 1.6A, i_2 \to -1.6A) \]

\[ w_{f\text{inal}} = \frac{1}{2} (5)(1.6)^2 + \frac{1}{2} (20)(-1.6)^2 = 32J \]

\[ w_R = \int_0^\infty p\,dt = \int_0^\infty \left( \frac{v(t)^2}{R_{eq}} \right)\,dt = \int_0^\infty \left( \frac{96e^{-2t}}{8} \right)\,dt = 288J \]

- Note \( w_R + w_{\text{final}} = w_{\text{init}} \)
- \( w_R \) indicates energy dissipated in resistors after switch opens
- \( w_{\text{final}} \) is energy retained by inductors due to the current circulating between the two inductors (+1.6A and -1.6A) when they become short circuits at steady state again

Figure: 07-08Ex7.2

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Natural Response of RC circuit

- Similar to that of an RL circuit
- Assume all currents and voltages in circuit have reached steady state (constant, dc) values

Prior to switch moving from a to b,
- C is acting as open circuit (i.e. since at DC)
- So all of $V_g$ appears across C since $I = 0$
- We want to find $v(t)$ for $t > 0$
- Note that since voltage across capacitor cannot change instantaneously, $V_g = V_0$, the initial voltage on capacitor

\[ C \frac{dv}{dt} + \frac{v}{R} = 0 \]

Solving,
\[ v(t) = v(0)e^{-\frac{t}{RC}} ; t \geq 0 \]
\[ v(0^+ - v(0) - v(0^+) = V_g = V_0 \]
\[ \tau = RC \]
\[ v(t) = V_0 e^{-\frac{t}{\tau}} ; t \geq 0 \]
Example 7.3

• To find $v_C(t)$ for $t \geq 0$, note that since cct is in steady state before switch moves from x to y, C is charged to 100V. The resistor network can be simplified with an equivalent 80k resistor.

• Simplify resistors with $R_{eq} = 32 + 240||60 = 80k\Omega$

• Then $\tau = RC = (0.5\mu F)(80k\Omega) = 40\text{ ms}$,

• voltage across 240 k\Omega and 60 k\Omega,

• current in 60 k\Omega resistor

• power dissipated in 60 k\Omega

• Energy dissipated in 60 k\Omega
Example 7.4: Series capacitors

- Initial voltages established by “hidden” sources

\[
\begin{align*}
\therefore v(t) &= 20e^{-t}V, t \geq 0 \\
i(t) &= \frac{v(t)}{250k\Omega} = 80e^{-t} \mu A, t \geq 0^+ \\
\therefore v_1(t) &= -\frac{1}{5\mu F} \int_0^t (80e^{-x} \mu A) dx - 4, t \geq 0 \\
v_1(t) &= (16e^{-t} - 20)V, t \geq 0 \\
v_2(t) &= -\frac{1}{20\mu F} \int_0^t (80e^{-x} \mu A) dx + 24, t \geq 0 \\
v_2(t) &= (4e^{-t} + 20)V, t \geq 0 \\
w_{C_1} &= 0.5CV^2 = 0.5(5\mu F)(4V)^2 = 40\mu J \\
w_{C_2} &= 0.5(20\mu F)(24V)^2 = 5760\mu J \\
w_0 &= 5800\mu J \\
t \to \infty, v_1 \to -20V, v_2 \to 20V \\
w_\infty &= 0.5(25\mu F)(20)^2 = 5000\mu J \\
w_{250k\Omega} &= \int_0^\infty p dt = \int_0^\infty \frac{(20e^{-t})^2}{250k\Omega} dt = 800\mu J
\end{align*}
\]
Step response of RL circuits

\[ KVL: V_s = iR + L \frac{di}{dt} \]
\[ \frac{di}{dt} = -Ri + \frac{V_s}{L} = -R \left( i - \frac{V_s}{R} \right) \]
\[ di = -\frac{R}{L} \left( i - \frac{V_s}{R} \right) dt \]
\[ i - \frac{V_s}{R} \]
\[ \int_{i_0}^{i(t)} dx = -\frac{R}{L} \int_0^t dy \]
\[ \ln \left( \frac{i(t) - \frac{V_s}{R}}{I_0 - \frac{V_s}{R}} \right) = -\frac{R}{L} t \]
\[ i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-R/L} \]

\[ v = V_s - \frac{R}{L} v(t) \]
\[ v = V_s e^{-(R/L)t} \]
Example 7.5: RL step response
Step response of RC circuits
Example 7.6: RC Step Response