Finite-element solution of the problem of scattering from cavities in metallic screens using the surface integral equation as a boundary constraint

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This work presents a novel finite-element solution to the problem of scattering from multiple two-dimensional cavities in infinite metallic walls. The technique presented here is highly efficient in terms of computing resources and is versatile and accurate in comparison with previously published methods. The formulation is based on using the surface integral equation with the free-space Green’s function as the boundary constraint. The solution space is divided into local bounded frames containing each cavity. The finite-element formulation is applied inside each frame to derive a linear system of equations associated with nodal field values. The surface integral equation is then applied at the opening of the cavities to truncate the computational domain and to connect the matrix subsystem generated from each cavity. The near and far fields are generated for different single and multiple cavity examples. The results are in close agreement with methods published earlier.

1. INTRODUCTION

Modeling of wave scattering from cavities in metallic surfaces and cavity-backed antennas has become increasingly important due to the inefficiency and inaccuracy of generic full-wave-field solvers. Furthermore, extensive recent interest in the problem of extraordinary transmission of light [1,2] and plasmonics resonance [3] has renewed interest in the solution of cavities and holes in conducting structures.

Grating couplers may consist of finite periodic subwavelength cavities or holes engraved in infinite-sized surfaces coated by conducting layers. An accurate calculation of near and far fields scattered from grating couplers allows manipulation and localization of light in novel applications such as near-field microscopy, surface defect detection, subwavelength lithography, tunable optical filters, and efficient solar cell devices.

Traditionally, the problem of scattering from a two-dimensional (2D) single cavity or hole in a perfect electric conductor (PEC) is solved by decoupling the fields inside the cavity or hole from those outside by closing the apertures with a PEC surface and introducing equivalent magnetic currents over the openings. Fields on the apertures are expressed by forcing the continuity condition on the tangential components of the fields. An appropriate Green’s function is required to express the fields due to equivalent magnetic currents inside and outside the cavity or hole. Among the earliest works in [4,5], the method of moments (MoM) is used to calculate the equivalent magnetic current on the apertures. The MoM formulation is represented by two n-port generalized networks. These networks are connected by current sources in parallel. Although using MoM is a powerful method to calculate the magnetic current on the apertures, deriving the Green’s function inside the cavity limits this method to canonical shapes and only homogenous and isotropic cavity fillings.

Standard impedance boundary condition (SIBC) and generalized impedance boundary condition (GIBC) are the other class of analytic methods used to approximate the fields inside the cavity [6,7]. Since SIBC and GIBC are suitable for simulating infinite planar dielectric coatings, these methods have limitation when encountering material discontinuities and cavity edges. Additionally, SIBC and GIBC are not applicable for multiple cavity structures.

Methods based on integral equations (IEs) are also used to solve the problem of scattering from cavities. Although these methods are applicable to general-shape cavities, their applications to cavities with infinite conducting walls involves geometry-specific approximations. Another problem that arises with the integral equation-based methods is the resultant dense system matrix, which is computationally expensive to solve. To overcome these deficiencies, hybrid methods based on the physical optics and method of moments (PO-MoM) was introduced in [8]. In the PO-MoM method, the induced current, which is the solution of the PO method for the unperturbed conductor sheet, is corrected with the MoM solution.

One of the powerful methods to solve the problem of scattering is the modal-based method. Extensive studies based on the mode matching technique have been re-
ported in the literature. Typically, for a single rectangular cavity engraved in a PEC surface, the fields inside the cavity are expressed in terms of Fourier series of the parallel plate waveguide modes. In the space exterior to the cavity, the scattered field is expressed in Fourier terms. Matching the modes inside and outside the cavity at the aperture determines the unknown modes’ coefficients. In [9], cylindrical Bessel function are used to expand the Fourier integral into series form for the cases of TE and TM polarization of incident Gaussian waves. In [10,11], the parallel plate waveguide modes are used to expand the Fourier integral. In [12], an extension to a finite number of cavities with identical dimensions and spacing into an infinite PEC screen was presented for the TM case. Multiple rectangular cavities with arbitrary dimension and spacing are investigated for both the TM and TE polarization in [13].

Recently, the mode matching technique based on the Fourier representation of the field was applied to solve the problem of scattering from a general-shape cavity [14]. In [14], the staircase approximation is used to divide the cavity into several rectangular layers. In each layer, the fields are expressed using the parallel-plate waveguide modes. Boundary conditions are applied to match the fields at the interface of the layers. The near and far fields are determined for the case of TM polarization. Extension of [14] to multiple general-shape cavities was presented in [15]. Although the modal method is very efficient and effective in solving the problem of homogeneously filled cavities, it cannot be used when encountering cavities having inhomogeneous or anisotropic fillings.

Differential equation methods such as the finite-element method (FEM) and the finite-difference time-domain method (FDTD) are suitable for the problem of scattering from general-shape cavities with complex fillings. In the FEM, the variational equation related to the weak form of Helmholtz equation is discretized. In the case of scatterers with finite geometry, the unbounded region around the target is truncated using an absorbing boundary condition (ABC) or perfectly matched layers (PML). In the case of infinite scatterers such as a cavity within an infinite ground plane, it is impossible to fully enclose the scatterer’s geometry (see Fig. 1). Therefore, the behavior of the scattered field due to the infinite PEC wall outside the computational domain boundary cannot be modeled properly. As shown in Fig. 1, a portion of the scattered field from the PEC wall propagates into the solution region, which causes errors. To minimize these errors, the domain truncating boundaries should be far enough from the cavity to enclose a larger segment of the PEC wall in addition to the cavity. However, placing the boundary of the computational domain far from the cavity leads to a prohibitive increase in the computational cost, in addition to inaccuracy in the solution due to the exclusion of a large part of the scatterer.

To address the shortcomings of the generic type of FEM solutions incorporating ABCs or PMLs, the boundary integral method (BIM) using the free-space Green’s function was introduced in [16,17]. In these works, the mesh region is truncated at the opening of the cavity or hole. The BIM is applied to derive an explicit form of a boundary condition of the third kind (mixed boundary condition) at the cavity opening [18]. Another FEM-based approach to solve the problem of scattering from unbounded geometries is enclosing the local sources, inhomogeneities, and anisotropies by an imaginary contour and using differential methods inside the contour to determine the field values [19]. The integral equation employing the free-space Green’s function is used as the boundary condition in [19]. Based on the method published in [19], both TM and TE scattering from a single rectangular cavity using the scattered field formulation were introduced in [20].

In this paper, we develop a new FEM-based method to solve the problem of scattering from multiple cavities in infinite metallic structures. The method presented here is an extension of the concept introduced earlier in [20] to multiple cavities using a total field formulation. The unbounded region in the computational domain is divided into bounded frames containing each cavity plus a thin layer above the cavity’s openings. Each layer is limited to the width of the cavity’s aperture (see Fig. 2). The FEM formulation is used to obtain the solution of the Helmholtz equation inside the local frames. The surface integral equation using the free-space Green’s function that was reported in [20] is applied at the opening of the groove as a global boundary condition. This boundary condition determines the behavior of nodes on the local frame boundary in terms of interior nodes. The Neumann or Dirichlet boundary condition is applied on the walls of the cavities in the TE or TM case, respectively.

We emphasize that in this work the formulation is based on the total field rather than the scattered field. In the scattered field formulation, the nonzero scattered field on the PEC wall must be taken into account when computing the surface integral. Since the integral domain is

![Fig. 1](image1.png)  
**Fig. 1.** (Color online) Schematic of the scattering problem from a cavity with arbitrary shape in an infinite PEC surface. An ABC or PML is used to truncate the computational domain.

![Fig. 2](image2.png)  
**Fig. 2.** (Color online) Schematic of the scattering problem from a cavity in a PEC surface. The dashed line represents the bounded region that contains all sources, inhomogeneities, and anisotropies.
infinite, in [20] an approximation was applied that took advantage of the limited support of the Green's function. If the scattered field formulation in [20] is extended to multiple cavities, the approximations used earlier deteriorate when the cavities are within distances smaller than the support of the Green’s function. In the total field formulation, however, the surface integration is limited to the aperture of the cavities, since it is zero on the PEC wall (after choosing the appropriate Green’s function). Therefore, the problem of multiple cavities does not pose any particular challenge under the total field formulation. Furthermore, the method presented here is applicable to both TM and TE polarization cases.

The organization of the paper is as follows. In Section 2, the FEM formulation of the problem for a single cavity is presented. In Sections 3 and 4, the boundary integral equations using the free-space Green’s function for the TM and TE cases are derived, respectively, to truncate the equations using the free-space Green’s function. In Section 5, numerical results are presented and validated using methods published in the literature and available commercial packages where possible.

2. FINITE-ELEMENT FORMULATION OF THE PROBLEM

Figure 2 shows a 2D cavity having an arbitrary shape in a PEC surface and illuminated by an obliquely incident plane wave. The angle $\phi_{inc}$ represents the angle of the incident wave, and $\mathbf{u}_{inc}$, $\mathbf{u}_{ref}$, and $\mathbf{u}^s$ denote the incident, reflected, and scattered fields along the cavity axis, respectively. Let $\Gamma_B$ represent the contour at the cavity opening and $\Gamma_O$ the top contour in the close vicinity of $\Gamma_B$. Also let $\Omega_n$ denote the interior region of the cavity including the layer between $\Gamma_B$ and $\Gamma_O$. We use conventional FEM formulation inside $\Omega_n$ to obtain the weak form of Helmholtz’s equation,

$$\nabla \cdot \left[ \frac{1}{p(x,y)} \nabla \mathbf{u}^t \right] + k_0^2 q(x,y) \mathbf{u}^t = g,$$  \hspace{1cm} (1)

where $\mathbf{u}^t$ is the total field. The time-harmonic factor $\exp(j\omega t)$ is assumed and suppressed throughout. $p(x,y)$ and $q(x,y)$ are defined as $\mu(x,y)$ and $\varepsilon(x,y)$, respectively, for the TM polarization, or $\varepsilon(x,y)$ and $\mu(x,y)$, respectively, for the TE polarization. $k_0$ is the propagation constant of the wave in free space. By defining the residual $r$ as

$$r = \nabla \cdot \left[ \frac{1}{p(x,y)} \nabla \mathbf{u}^t \right] + k_0^2 q(x,y) \mathbf{u}^t - g$$ \hspace{1cm} (2)

and using the weighting function $w_i$, the weighted residual integral $R_i$ is defined as

$$R_i = \int_{\Omega_n} w_i \left[ \nabla \cdot \left[ \frac{1}{p(x,y)} \nabla \mathbf{u}^t \right] + k_0^2 q(x,y) \mathbf{u}^t - g \right] \, d\Omega = 0.$$ \hspace{1cm} (3)

Using Green’s first identity, Eq. (3) can be rewritten as

$$R_i = -\int_{\Omega_n} \left[ \frac{1}{p(x,y)} \nabla w_i \cdot \nabla \mathbf{u}^t - k_0^2 q(x,y) w_i \mathbf{u}^t + g w_i \right] \, d\Omega + \oint_{\Gamma_B} w_i \nabla \mathbf{u}^t \cdot d\Gamma = 0.$$ \hspace{1cm} (4)

Following the method reported in [19], the FEM formulation is applied only inside the solution region. Inside the solution domain, $R_i$ is reduced to

$$R_i = -\int_{\Omega_n} \left[ \frac{1}{p(x,y)} \nabla w_i \cdot \nabla \mathbf{u}^t - k_0^2 q(x,y) w_i \mathbf{u}^t + g w_i \right] \, d\Omega = 0.$$ \hspace{1cm} (5)

Next, the solution domain $\Omega_n$ is discretized into triangular elements. The unknown field over each element is described by a set of interpolating functions given by

$$\mathbf{u}^t = \sum_{i=1}^{m} \alpha_i(x,y),$$ \hspace{1cm} (6)

where $m$ is number of nodes at which the unknown field is defined, and $\alpha_i(x,y)$ is the interpolation function. By choosing $w_i = \alpha_i(x,y)$—Galerkin’s method—$R$ can be expressed in matrix form as

$$R = [M][U] - [F] = 0,$$ \hspace{1cm} (7)

where $U$ represents the unknown field value at each node. The elements of $m \times m$ matrix $M$ and $m \times 1$ matrix $F$ are given by

$$M_{ij} = \int_{\text{Element}} \left[ \frac{1}{p(x,y)} \nabla \alpha_i(x,y) \cdot \nabla \alpha_j(x,y) - k_0^2 q(x,y) \alpha_i(x,y) \alpha_j(x,y) \right] \, d\Omega,$$

and

$$F_i = -\int_{\text{Element}} g \alpha_i(x,y) \, d\Omega.$$ \hspace{1cm} (8)

Equation (7) can be represented symbolically as

$$\begin{bmatrix} M_{ii} & M_{ib} & 0 \\ M_{bi} & M_{bb} & M_{bo} \\ 0 & M_{ob} & M_{oo} \end{bmatrix} \begin{bmatrix} u_i \\ u_b \\ u_o \end{bmatrix} = \begin{bmatrix} F_i \\ F_b \\ F_o \end{bmatrix},$$ \hspace{1cm} (9)

where $u_i$, $u_b$, and $u_o$ represent nodal field values inside the cavity, on $\Gamma_B$, and on $\Gamma_O$, respectively. The $[F]$ matrix represents impressed sources at each node; therefore $[F]$ is zero in this problem.

The linear system of equations in Eq. (9) represents the relationship between the nodal field values without any external constraint. The imposition of a specific excitation represented by the incident plane wave has to be taken into consideration through a boundary constraint that establishes a relationship between the incident field, the boundary nodes, and the interior nodes. In Section 3, the surface integral equation will be developed and used as a boundary constraint in lieu of the last term in Eq. (4), which cannot be implemented directly since the field on $\Gamma_B$ is not known.
3. SURFACE INTEGRAL EQUATION FOR TM POLARIZATION

The surface integral equation using the free-space Green’s function will be used to express the nodal field values on $\Gamma_0$ in terms of the nodal field values on $\Gamma_\infty$. Let us consider the domain $\Omega_\infty$ representing the half-space above the PEC (see Fig. 3). In $\Omega_\infty$ and for the TM polarization case, the electric field vector has only a $z$-component satisfying the Helmholtz equation,

$$\nabla^2E_z(p) + k_0^2E_z(p) = j\omega\mu J_z(p), \quad p \in \Omega_\infty,$$

where $J_z(p)$ is an electric current inside $\Omega_\infty$. Let us introduce Green’s function $G^\parallel(p, p')$ that is the solution due to an electric current filament located at $p'$ and governed by the Helmholtz equation,

$$\nabla^2G^\parallel(p, p') + k_0^2G^\parallel(p, p') = -\delta(p-p'), \quad p, p' \in \Omega_\infty.$$

(11)

$G^\parallel(p, p')$ satisfies the boundary condition $G^\parallel(p, p')\big|_{\Gamma_\infty}=0$ (i.e., $G^\parallel=0$ on $\Gamma$) and the Sommerfeld radiation condition at infinity. $G^\parallel(p, p')$ is easily found to be the zeroth-order Hankel function of the second kind:

$$G^\parallel(p, p') = -\frac{j}{4}H_0^0(|p-p'|) + \frac{j}{4}H_0^0(|p-p'|).$$

(12)

Multiplying both sides of Eq. (10) by $G^\parallel(p, p')$ and integrating over $\Omega_\infty$ yields

$$\int_{\Omega_\infty} G^\parallel(p, p')[\nabla^2E_z(p) + k_0^2E_z(p)]d\Omega = j\omega\mu \int_{\Omega_\infty} J_z(p)G^\parallel(p, p')d\Omega;$$

(13)

invoking Green’s second identity,

$$\int\nabla E_z \nabla G^\parallel - G^\parallel \nabla^2 E_z d\Omega = \oint_{\Gamma \cup \Gamma_\infty} \left( E_z \frac{\partial G^\parallel}{\partial n} - G^\parallel \frac{\partial E_z}{\partial n} \right) d\Gamma,$$

(14)

where $\Gamma + \Gamma_\infty$ is the contour enclosing $\Omega_\infty$, Eq. (13) can be written as

$$\int_{\Omega_\infty} E_z(p)(\nabla^2G^\parallel + k_0^2G^\parallel)d\Omega = j\omega\mu \int_{\Omega_\infty} J_z(p)G^\parallel(p, p')d\Omega + \oint_{\Gamma \cup \Gamma_\infty} \left( E_z \frac{\partial G^\parallel}{\partial n} - G^\parallel \frac{\partial E_z}{\partial n} \right) d\Gamma.$$  

(15)

Substituting Eq. (11) into Eq. (15), we have

$$E_z(p') = -j\omega\mu \int_{\Omega_\infty} J_z(p)G^\parallel(p, p')d\Omega - \oint_{\Gamma \cup \Gamma_\infty} \left[ E_z(p) \frac{\partial G^\parallel}{\partial n} - G^\parallel(p, p') \frac{\partial E_z}{\partial n} \right] d\Gamma.$$  

(16)

Since both $E_z$ and $G^\parallel$ satisfy the Sommerfeld radiation condition at infinity, integration over $\Gamma_\infty$ (see Fig. 3) on the right-hand side of Eq. (16) vanishes. Note that $G^\parallel$ is zero on $\Gamma$ [see Eq. (12)]. Additionally, $E_z(p)$ is zero over the PEC ground plane except on the cavity aperture. On interchanging primed and unprimed coordinates, Eq. (16) can be simplified to

$$E_z(p) = -j\omega\mu \int_{\Omega_\infty} J_z(p')G^\parallel(p, p')d\Omega - \int_{\text{Aperture}} E_z(p') \frac{\partial G^\parallel(p, p')}{\partial n'} d\Gamma.$$  

(17)

In Eq. (17), the first term on the right-hand side represents the electric field generated by the current filament and its image in the vicinity of the PEC ground plane. We can interpret these fields as incident and reflected waves. The second term in Eq. (17) represents the field perturbation due to the cavity. In other words, the total electric field at each point in the upper half-space is the sum of the incident field, the reflected field due to the cavity, and the scattered field due to the aperture of the cavity [18] as

$$E_z(p) = E_z^{\text{inc}}(p) + E_z^{\text{ref}}(p) = \int_{\text{Aperture}} E_z(p') \frac{\partial G^\parallel(p, p')}{\partial n'} d\Gamma.$$  

(18)

Referring to Fig. 2, let $p$ and $p'$ be designated the position of nodes on $\Gamma_0$ and $\Gamma_B$, respectively. Therefore the incident and the reflected waves can be written as

$$E_z^{\text{inc}} = \exp[jk(x \sin \theta - y \cos \theta)],$$

$$E_z^{\text{ref}} = -\exp[jk(x \sin \theta + y \cos \theta)],$$

(19)

where $x$ and $y$ are Cartesian components of $p$. To calculate the last term in Eq. (18), the aperture $\Gamma_B$ is discretized into $n$ segments with length of $\Delta x'$. We then expand $E_z(p')$ over $\Gamma_\infty$ in terms of piecewise linear interpolating functions as
\[ E_x(p') = \sum_{j=1}^{n} E_j \sum_{k=1}^{2} \psi_{jk}(x'_j), \]  

where \( x' \) and \( y' \) are Cartesian components of \( p' \) and

\[ \psi_{jk}(x'_j) = \begin{cases} \frac{x'_j}{\Delta x'}, & k = 1 \\ \frac{1 - x'_j}{\Delta x'}, & k = 2 \end{cases}. \]  

Equation (18) can be represented in matrix notation as

\[ [u_a] = [T] + [S]u_b, \]  

where the elements of \([u_a], [u_b], \) and \([T]\) matrices represent \( E_x(p), E_x, \) and \((E_{\text{inc}}^x + E_{\text{ref}}^x)\), respectively, at each node. The elements of \([S]\) are defined as

\[ S_{ij} = \int_{x'_j - \Delta x'}^{x'_j + \Delta x'} \psi_{i}(x'_j) \frac{\partial G^x_{ij}}{\partial y'} \, dx' + \int_{x'_j - \Delta x'}^{x'_j + \Delta x'} \psi_{j}(x'_j) \frac{\partial G^x_{ij}}{\partial y'} \, dx'. \]  

Equation (22) represents the boundary condition on the cavity opening, which is different from that of (18). Combining Eq. (9) and Eq. (22) in matrix form results in the reduced system matrix

\[ \begin{bmatrix} M_{ij} & M_{ib} \\ M_{bi} & M_{bb} + M_{bo}S \end{bmatrix} \begin{bmatrix} u_i \\ u_b \end{bmatrix} = \begin{bmatrix} F_i + M_{bo}T \end{bmatrix}. \]  

Equation (24) represents the modified system matrix, which can be solved using standard methods.

### 4. SURFACE INTEGRAL EQUATION FOR TE POLARIZATION

To derive the surface integral equation for the TE polarization case, we replace the electric current filament with a magnetic current filament \( M_z \). In \( \Omega_\text{s} \) and for the TE polarization case, the magnetic field vector has only a \( z \)-component satisfying the Helmholtz equation,

\[ \nabla^2 H_z(p) + k_0^2 H_z(p) = j \omega M_z(p), \quad p \in \Omega_s, \]  

where \( p \) and \( \Omega_s \) have the same definition as in Section 3 (Fig. 3). Let us introduce the Green’s function \( G^b(p, p') \) that is the solution due to the magnetic current filament located at \( p' \) and governed by the Helmholtz equation:

\[ \nabla^2 G^b(p, p') + k_0^2 G^b(p, p') = -\delta(p-p'), \quad p, p' \in \Omega_s. \]  

Since the image of the magnetic current in the vicinity of PEC is in the same direction as the original current, \( G^b(p, p')_{y=0} \neq 0 \) on the ground plane. In this case, the boundary condition on the PEC surface is

\[ \left. \frac{\partial G^b(p, p')}{\partial y'} \right|_{y=0} = 0. \]  

In addition, \( G^b(p, p') \) satisfies the Sommerfeld radiation condition at infinity. Therefore \( G^b(p, p') \) can be represented in terms of the zeroth-order Hankel function of the second kind as

\[ G^b(p, p') = \frac{j}{4} H_0^2(|p-p'|_\text{source}) - \frac{j}{4} H_0^2(|k|p-p'|_\text{image source}). \]  

Multiplying both sides of Eq. (25) by \( G^b(p, p') \) and following a procedure similar to that of the TM polarization case, we obtain

\[ H_z(p') = -j \omega \int_{\Omega_s} M_z(p) G^b(p, p') \, d\Omega \]  

\[ - \int_{\Gamma} \left[ H_z(p) \frac{\partial G^b(p, p')}{\partial n} - G^b(p, p') \frac{\partial H_z(p)}{\partial n} \right] \, d\Gamma. \]  

Both \( H_z(p) \) and \( G^b(p, p') \) satisfy the Sommerfeld radiation condition at infinity; therefore integration over \( \Gamma' \) in the right-hand side of Eq. (29) vanishes. Note that \( \frac{\partial H_z(p)}{\partial n} \) is zero over the PEC ground plane except on the cavity apertures, and \( \frac{\partial G^b(p, p')}{\partial n} \) is zero on the \( \Gamma' \) [see Eq. (27)]. On interchanging primed and unprimed coordinates, Eq. (29) is reduced to

\[ H_z(p) = -j \omega \int_{\Omega_s} M_z(p) G^b(p, p') \, d\Omega \]  

\[ + \int_{\text{Aperture}} G^b(p, p') \frac{\partial H_z(p')}{\partial n} \, d\Gamma. \]  

As in the TM case, the first term on the right-hand side of Eq. (30) represents the magnetic field generated by the current filament and its image in the vicinity of the PEC ground plane. We can interpret these fields as incident and reflected waves [18]. The second term in Eq. (30) represents field perturbation due to the cavity. Then Eq. (30) can be rewritten as

\[ H_z(p) = H_z^\text{inc}(p) + H_z^\text{ref}(p) + \int_{\text{Aperture}} G^b(p, p') \frac{\partial H_z(p')}{\partial n'} \, d\Gamma. \]  

By the same definition of \( p \) and \( p' \) and assuming the coordinate system as in Fig. 2, the incident and reflected waves can be written as

\[ H_z^\text{inc} = \exp[jk(x \sin \theta - y \cos \theta)], \]  

\[ H_z^\text{ref} = \exp[jk(x \sin \theta + y \cos \theta)]. \]
To calculate the last term in Eq. (31), the partial derivative \( \frac{\partial H_z(x')}{\partial n'} \) can be conveniently expressed as a first-order finite difference by

\[
\frac{\partial H_z(x')}{\partial n'} = - \frac{H_z(x=x',y') - H_z(x',y')}{y - y'}
\]

Equation (33) can be rearranged as

\[
S_{ij} = \int_{\Delta x} G(x,y,x',y') \psi(x',y') dx',
\]

Eq. (31) can be represented in matrix form as

\[
[u_b] = [T] - [S][u_a] - [u_b],
\]

where the elements of \([u_a]\), \([u_b]\), and \([T]\) matrices represent \(H_z(x,y), H_z(x',y'),\) and \(H^t_z(x,y) + H^t_z(x',y')\), respectively, at each node. Equation (37) can be rearranged as

\[
[u_a] = \{1 + [S]^{-1}[T]\} + \{1 + [S]^{-1}[S][u_a].
\]

Equation (38) represents the boundary condition on the cavity opening. Combining Eq. (9) and Eq. (38) in matrix form results in the reduced system matrix

\[
[M_{bb}] + [M_{bb}(1 + S)^{-1}S][u_b] = \{F_b - M_{bo}(1 + S)^{-1}T\}.
\]

Equation (39) represents the modified system matrix, which can be solved using standard methods.

5. EXTENSION TO MULTIPLE CAVITIES

In this section, we extend the method developed above to the problem of scattering from multiple arbitrary-shape cavities in a PEC surface. A schematic representing the scattering problem for two cavities is shown in Fig. 4. Extending the PEM in Section 2 to two cavities, we generalize the system matrix for the domains \(\Omega_1\) and \(\Omega_2\) as

\[
[M_{1}]^{(1)}[u_{1}]^{(1)} = [F]^{(1)},
\]

where each system of equation can be represented symbolically as Eq. (9). Assembling the two systems using global numbering of nodes gives

\[
[M_{1}]^{(1)}[u_{1}]^{(1)} = [F]^{(1)}.
\]

The two matrix systems arising from each of the two cavities will be coupled through the surface integral equation in the following manner. In Eq. (18) and Eq. (31), each node on \(\Gamma_o\) is connected via the Green’s function to all the nodes on the aperture of the cavities \(\Gamma_b\) (see Fig. 4). In other words, the cavities are coupled to each other only through the surface integral equation and the Green’s function. In Eq. (18) and Eq. (31), the integration is performed over the apertures of all cavities. For instance, Eq. (18) for the TM polarization can be represented symbolically in matrix form as

\[
[M_{1}]^{(1)}[u_{1}]^{(1)} = [F]^{(1)},
\]

where \([S]^{(i)}\) represents connectivity between nodes on \(\Gamma_o\) of the \(i\)th cavity \([u_a]^{(i)}\) and nodes on \(\Gamma_b\) of the \(j\)th cavity \([u_b]^{(j)}\) via the surface integral equation \((i,j=1,2)\). Combining Eq. (42) and Eq. (41) in matrix form results in the reduced system matrix

\[
[M_{1}]^{(1)}[u_{1}]^{(1)} = [F]^{(1)},
\]
Integration over the apertures of all the cavities results in addition of off-diagonal submatrices. The resultant system matrix becomes

\[
\begin{bmatrix}
[M']^{(1)} & [C]^{(12)} & \cdots & [C]^{(1N)} \\
[C]^{(21)} & [M']^{(2)} & \cdots & [C]^{(2N)} \\
\vdots & \vdots & \ddots & \vdots \\
[C]^{(N1)} & [C]^{(N2)} & \cdots & [M']^{(N)}
\end{bmatrix}
\begin{bmatrix}
[u']^{(1)} \\
[u']^{(2)} \\
\vdots \\
[u']^{(N)}
\end{bmatrix} =
\begin{bmatrix}
[F']^{(1)} \\
[F']^{(2)} \\
\vdots \\
[F']^{(N)}
\end{bmatrix},
\]  

(44)

where \([C]^{(12)}\) and \([C]^{(21)}\) are matrices representing the coupling between the two cavities and are given by

\[
[C]^{(12)} =
\begin{bmatrix}
0 & 0 \\
0 & [M_{bb}]^{-1}[S]^{(12)}
\end{bmatrix},
\]

\[
[C]^{(21)} =
\begin{bmatrix}
0 & 0 \\
0 & [M_{bb}]^{-2}[S]^{(21)}
\end{bmatrix},
\]  

(45)

and \([M']\), \([F']\), and \([u']\) are given by

\[
[M']^{(1)} =
\begin{bmatrix}
[M_{aa}]^{(1)} & [M_{ba}]^{(1)} \\
[M_{ba}]^{(1)} & [M_{bb}]^{(1)} + [M_{bb}]^{-1}[S]^{(11)}
\end{bmatrix},
\]

\[
[M']^{(2)} =
\begin{bmatrix}
[M_{aa}]^{(2)} & [M_{ba}]^{(2)} \\
[M_{ba}]^{(2)} & [M_{bb}]^{(2)} + [M_{bb}]^{-2}[S]^{(22)}
\end{bmatrix},
\]

\[
[F']^{(1)} =
\begin{bmatrix}
[F_b]^{(1)} - [M_{bb}]^{-1}[T]^{(1)} \\
[F_b]^{(2)} - [M_{bb}]^{-2}[T]^{(2)}
\end{bmatrix},
\]

\[
[F']^{(2)} =
\begin{bmatrix}
[F_b]^{(1)} - [M_{bb}]^{-1}[T]^{(1)} \\
[F_b]^{(2)} - [M_{bb}]^{-2}[T]^{(2)}
\end{bmatrix},
\]

and

\[
[u']^{(k)} =
\begin{bmatrix}
[u_a]^{(k)} \\
[u_b]^{(k)}
\end{bmatrix}, \quad (k = 1, 2).
\]  

(46)

Note that these coupling matrices are not necessarily identical. The same procedure is applicable to the TE polarization. Generalizing the formulation to \(N\) cavities results in the matrix system

\[
\begin{bmatrix}
[M']^{(1)} & [C]^{(12)} & \cdots & [C]^{(1N)} \\
[C]^{(21)} & [M']^{(2)} & \cdots & [C]^{(2N)} \\
\vdots & \vdots & \ddots & \vdots \\
[C]^{(N1)} & [C]^{(N2)} & \cdots & [M']^{(N)}
\end{bmatrix}
\begin{bmatrix}
[u']^{(1)} \\
[u']^{(2)} \\
\vdots \\
[u']^{(N)}
\end{bmatrix} =
\begin{bmatrix}
[F']^{(1)} \\
[F']^{(2)} \\
\vdots \\
[F']^{(N)}
\end{bmatrix},
\]  

(47)

Once the nodal field values on \(\Gamma_O\) are expressed in terms of the nodal field values on \(\Gamma_B\), the reduced system of equations can be solved using standard methods.

The formulation presented in this work is applicable to cavities present in PEC surfaces. For multiple cavities, Eqs. (44) and (47) give a mathematical quantification of the coupling factors between the cavities. Physically, we expect the cavities to be coupled through surface currents existing on the segments connecting the cavities, as that is the only mechanism for energy transfer between the cavities. In the numerical solutions in Section 6, we show the dependence of the surface currents on the distance between the cavities.

6. NUMERICAL RESULTS

Once the system of linear equations—Eq. (24) for the TM polarization or Eq. (39) for the TE polarization—is derived, its solution can be obtained using standard methods. In this work, we use the conjugate gradient method (CGM). The reason that we prefer the CGM to the other methods is that in CGM the roundoff error is not accumulated over the iterations. In this section, we provide examples of different cavities and make comparison with results using methods published earlier. Without loss of generality, the magnitude of the incident electric field is assumed to be unity throughout this work. To discretize the solution domain, we use first-order triangle elements with mesh density of approximately 20 nodes per \(\lambda\) for the TM case. Since there is a singularity in the electric field at the edges of the cavity in the TE case, we use mesh density of 100 nodes per \(\lambda\).

A. Single Cavity Case

In the first example, we considered a \(1\lambda \times 1.5\lambda\) (width \(\times\) depth) rectangular cavity, where \(\lambda\) is the wavelength in free space. Figures 5 and 6 show the total electric field at the aperture of the cavity for a TM incident plane wave, for normal and oblique incident angles. Comparison is made with the results published in [14] using the mode matching method as well as the result obtained using the commercial simulator HFSS [21]. Throughout this work, the solution obtained using the method presented in this paper is referred to as FEM-TFSIE.

The results in Figs. 5 and 6 show strong agreement between the calculations using FEM-TFSIE and those obtained using HFSS and the mode matching method. In the case of HFSS, the required computational domain was approximately \(10\lambda^2\) using 19 444 tetrahedral elements and requiring 174 s on a Pentium 2.4 GHz Core 2Quad CPU. (Note that the HFSS solution requires a 3D model with periodic boundary conditions to emulate 2D structures; therefore, a more meaningful comparison would be to use a FEM solver that employs an ABC or PLM to truncate the open computational space above the aperture.) This HFSS computational domain was needed so that solution would converge to the mode matching solution. On the other hand, the FEM-TFSIE solution space was confined...
to the cavity area of $1.5\lambda^2$ with a solution time of less than 1 s on the same computer platform.

In the second example, we considered a $0.6\lambda \times 0.4\lambda$ rectangular cavity. Figure 7 shows the total magnetic field at the aperture of the cavity for a TE impinging wave at oblique incident angle. The comparison is made with the results from [18] and those of HFSS. We observe that our method is in good agreement with both methods despite a small deviation between our results and those of HFSS; more specifically, at the edges of the cavity where the electric field has a singularity. The reason for this deviation is that we used triangular elements with a constant size since our meshing scheme is performed without an automated mesh generator. If an adaptive mesh scheme is employed as was the case in HFSS, the deviation in the field at the singularity location can be eliminated.

In the next example, we consider a cavity with the shape of a right triangle having an aperture of $1\lambda$. Figure 8 shows the aperture fields for the case of oblique incidence. Strong agreement between FEM-TFSIE and HFSS is observed.

As the final example of a single cavity case, we consider a $0.7\lambda \times 0.35\lambda$ rectangular cavity shown in Fig. 9. The cavity is filled with silicon having a relative permittivity of $\varepsilon_r = 11.9$ and including a $0.42\lambda \times 0.07\lambda$ PEC strip positioned at the geometric center of the cavity. The electric field at the aperture for the TM case is calculated. This numerical example shows the versatility of FEM-TFSIE in solving the problem of cavities with complex structures and fillings.

### B. Multiple Cavities

We consider two identical $1\lambda \times 0.5\lambda$ rectangular cavities separated by $0.2\lambda$. Figures 10 and 11 show the total electric field at the aperture of the two cavities, for normal and oblique incident plane waves with TM polarization. Close agreement between FEM-TFSIE and HFSS is observed.
The cavities are separated by 0.2λ.

Figure 12 shows the total magnetic field at the aperture of two identical 0.4λ × 0.2α rectangular cavities separated by 0.2α, TE case, for normal incidence. We observe strong agreement between FEM-TFSIE and HFSS, aside from small deviations at the edges of the cavities due to field singularities there.

Using the coordinate system as in Fig. 4, the surface current can be calculated as

\[ J(x) = \hat{z} \frac{1}{j \omega \mu} \frac{\partial E_y}{\partial y} \]  

(48)

for TM polarization, and

\[ J(x) = \hat{y} \times H_z = \hat{z} H_z \]  

(49)

for TE polarization. In Eqs. (48) and (49), \( E_y \) and \( H_z \) represent the total fields. Figures 13 and 14 show the magnitude of surface current for normal and oblique TM-polarized incident waves. It is observed that the induced current on the PEC surface increases as the cavity separation decreases. The magnitude of the incident field and the reflected field on the PEC surface are the same for both closely separated and distanced cavities. Thus, the change in surface current is due to the scattered field from the cavities. Since the scattered field affects the field distribution on the other cavities, we can interpret the magnitude of the surface current as a coupling factor between the cavities.

Finally, we considered six identical cavities in a PEC surface. The cavities are rectangular with dimension of 0.8λ × 0.4α and are separated by distances of 0.8λ. In this example, we present the field solution at the opening of the cavities and the far field. The far field can be calculated using the equivalence theorem. By closing the aperture by a PEC surface and introducing an equivalent magnetic current at the aperture \( \mathcal{M} \), the electric field at the far region can be represented as

\[ E(x,y) = -2 \nabla \times \int_{\text{Aperture}} \mathcal{M}(x',y') G(x,y,x',y') \, dx' \]  

(50)

where \( \mathcal{M}(x',y') = -\hat{n} \times E(x',y') \big|_{y'=0} \) and \( G(x,y,x',y') \) is defined by Eq. (12). For the TM case where the electric field has only a \( z \)-component, Eq. (50) can be written as

\[ E(x,y) = -2 \int_{\text{Aperture}} 2E_z(x',y') \big|_{y'=0} \frac{\partial G}{\partial y} \, dx' \]  

(51)

For the TE case, where the magnetic field has only a \( z \)-component, the electric field at the aperture has two components. Therefore Eq. (50) can be written as
methods presented in the literature. Furthermore, the FEM-TFSIE is very versatile in handling complex-shape cavities with inhomogeneous fillings without any modification to the algorithm. The run time and solution efficiency of FEM-TFSIE are two major attractive features of this method, making it well suited for optimization problems involving scattering from gratings in metallic screens.

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