Electromagnetic scattering from multiple sub-wavelength apertures in metallic screens using the surface integral equation method

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This work presents a novel finite-element solution to the problem of scattering from multiple two-dimensional holes with side grating in infinite metallic walls. The formulation is based on using the surface integral equation with free-space Green’s function as the boundary constraint. The solution region is divided into interior regions containing each hole or cavity as a side grating and exterior region. The finite-element formulation is applied inside the interior regions to derive a linear system of equations associated with nodal field values. The surface integral equation is then applied at the opening of the holes as a boundary constraint to connect nodes on the boundaries to interior nodes. The technique presented here is highly efficient in terms of computing resources, versatile and accurate in comparison with previously published methods. The near and far fields are generated for different single and multiple hole examples. © 2010 Optical Society of America

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1. INTRODUCTION

Extensive recent research in the problem of extraordinary transmission of light through sub-wavelength apertures [1,2] and plasmonic resonance [3], has renewed interest in modeling of wave scattering from grating surfaces. The inefficiency and inaccuracy of the generic full-wave solvers, however, has renewed attention to the efficient solution of the problem of scattering from cavities and holes engraved in conducting structures.

Grating couplers may consist of finite periodic sub-wavelength cavities or holes engraved in infinite-sized surfaces coated by conducting layers. An accurate calculation for near and far fields scattered from grating couplers allows manipulation and localization of light in novel applications such as near-field microscopy, surface defect detection, sub-wavelength lithography, developing tunable optical filters, and improving the efficiency of solar cell devices.

The problem of scattering from a two-dimensional (2-D) single hole in a perfect electric conductor (PEC) can be solved by decoupling the fields inside of the hole from outside by closing the apertures with a PEC surface and introducing equivalent magnetic currents over the openings in [4]. Fields on the apertures were expressed by forcing the continuity condition on the tangential components of the fields. An appropriate Green’s function was derived to express the fields due to equivalent magnetic currents inside and outside the hole. The Method of Moment (MoM) was used to calculate the equivalent magnetic current on the apertures in [4]. Although using MoM is a powerful method to calculate the magnetic current on the apertures, deriving the Green’s function inside the hole limits this method to canonical shapes and only homogenous and isotropic hole fillings.

The modal-based method was used to solve the problem of scattering from a single hole in [5,6]. The fields inside the hole were expressed in terms of Fourier series of the parallel-plate waveguide modes. In the space exterior to the hole, the scattered field was expressed in Fourier terms. Matching the modes inside and outside the hole at the apertures determined the unknown modes’ coefficients. The same method was used to solve the problem of scattering from multiple slits in parallel-plate waveguide [7]. Although the modal method is very efficient and effective in solving the problem of scattering from homogeneously filled holes, it cannot be used when encountering holes having inhomogeneous or anisotropic fillings.

Methods based on numerical approaches to solve the boundary value problems such as the finite element method (FEM) and the finite-difference time-domain method (FDTD) are suitable for the problem of scattering from general-shape holes with complex fillings. In the case of infinite scatterers such as a hole within an infinite ground plane, it is impossible to fully enclose the scatterer’s geometry by an absorbing boundary condition (ABC) or perfectly matched layers (PMLs) to truncate the solution region (see Fig. 1). Therefore, the behavior of the scattered field due to the infinite PEC wall outside the computational domain boundary cannot be modeled properly. As shown in Fig. 1, a portion of the scattered field from the PEC wall propagates into the solution region, which causes errors. To minimize these errors, the domain truncating boundaries should be far enough from the hole to enclose a larger segment of the PEC wall in
### 2. Finite-Element Formulation of the Problem

Figure 2 shows a 2-D hole having an arbitrary shape in a perfect-conductor surface and illuminated by an obliquely incident plane wave. The angle $\theta^{inc}$ represents the angle of the incident wave, and $u^{inc}$, $u^{ref}$, $u^s$, and $u^{trans}$ denote the incident, reflected, scattered, and transmitted fields along the hole axis, respectively. Next, we divide the problem space into three regions. Region $I$ and $II$ denote the upper and lower half-spaces of the PEC slab. Region $III$ represents the inside of the hole. Let $\Gamma_B^I$ and $\Gamma_B^II$ represent the contour at the interface of the openings of the hole with region $I$ and $II$, respectively. Also let $\Gamma_B^O$ and $\Gamma_B^O'$ be exterior to the hole and in close proximity to $\Gamma_B^I$ and $\Gamma_B^II$, respectively as shown in Fig. 2. Let $\Omega_{in}$ denote the interior region of the hole, region $III$, including the layer between $\Gamma_B$ and $\Gamma_O$ in the region $I$ and $II$. We use the finite-element formulation inside $\Omega_{in}$ to obtain the weak form of Helmholtz’s equation:

$$\nabla \cdot \left( \frac{1}{p(x,y)} \nabla u^i \right) + k_0^2 q(x,y) u^i = g,$$

where $u^i$ is the total field and $g$ is the impressed source. The time harmonic factor $\exp(\text{io}t)$ is assumed and suppressed throughout. $p(x,y)$ and $q(x,y)$ are defined as $\mu_r(x,y)$ and $\varepsilon_r(x,y)$, respectively, for TM polarization, or $\varepsilon_r(x,y)$ and $\mu_r(x,y)$, respectively, for TE polarization. $k_0$ is the propagation constant of the wave in free space. The weighted residual $R_i$ is defined as

$$R_i = \int_{\Omega_{in}} \left( \nabla \cdot \left( \frac{1}{p(x,y)} \nabla u^i \right) + k_0^2 q(x,y) u^i - g \right) \cdot d\Omega = 0,$$

where $w_i$ is the weighting function. Using Green’s first identity, Eq. (2) can be rewritten as

$$R_i = -\int_{\Omega_{in}} \left( \frac{1}{p(x,y)} \nabla w_i \cdot \nabla u^i - k_0^2 q(x,y) w_i u^i + g w_i \right) \cdot d\Omega + \oint_{\Gamma_B^I} \left( \frac{w_i}{p(x,y)} \nabla u^i \cdot d\Gamma \right) = 0.$$

Following the method reported in [11], the finite element
formulation is applied only inside the solution region. Inside the solution domain, \( R_i \) is reduced to

\[
R_i = - \int_{\Omega_m} \left( \frac{1}{p(x,y)} \nabla w_i \cdot \nabla u' - k_0^2 q(x,y) w_i u' + gw_i \right) d\Omega = 0.
\]

(4)

Next, the solution domain \( \Omega_m \) is discretized into triangular elements. The unknown field over each element is described by a set of interpolating functions given by

\[
u' = \sum_{i=1}^{m} u'_i \alpha_i(x,y), \quad i = 1, 2, 3, \ldots, m,
\]

(5)

where \( m \) is the number of nodes at which the unknown field is defined, and \( \alpha_i(x,y) \) are interpolation functions. \( u'_i \) represents unknown field values at each node. Using Galerkin's method, we set \( w_i = \alpha_i(x,y) \). Then, \( R \) can be expressed in matrix form,

\[
R = [M][U] - [F] = 0,
\]

(6)

where \( U \) represents the unknown field value at each node. The elements of the \( m \times m \) matrix \( M \) and the \( m \times 1 \) matrix \( F \) are given by

\[
M_{ij} = \int_{\text{Element}} \left( \frac{1}{p(x,y)} \nabla \alpha_i(x,y) \cdot \nabla \alpha_j(x,y) - k_0^2 q(x,y) \alpha_i(x,y) \alpha_j(x,y) \right) d\Omega,
\]

\[
F_i = - \int_{\text{Element}} g \alpha_i(x,y) d\Omega.
\]

(7)

Equation (6) can be represented symbolically as

\[
\begin{bmatrix}
M_{ii} & M_{ib} & M_{bi} & 0 & 0 \\
M_{bi} & M_{bb} & 0 & M_{b\alpha} & 0 \\
M_{bi} & 0 & M_{bb} & 0 & M_{b\alpha} \\
0 & M_{b\alpha} & 0 & M_{\alpha\beta} & 0 \\
0 & 0 & M_{\alpha\beta} & 0 & M_{\alpha\gamma}
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_{b1} \\
u_{b2} \\
u_{\alpha1} \\
u_{\alpha2}
\end{bmatrix}
= \begin{bmatrix}
F_i \\
F_{b1} \\
F_{b2} \\
F_{\alpha1} \\
F_{\alpha2}
\end{bmatrix},
\]

(8)

where \( u_i, u_{b1}, u_{b2}, u_{\alpha1}, u_{\alpha2} \) represent the nodal field values inside the hole on \( \Gamma_H^I \) and on \( \Gamma_H^{1/2} \), respectively. The \([F]\) matrix represents any possible impressed sources at each node; therefore, \([F]\) is zero in this problem.

The linear system of equations in Eq. (8) represents the relationship between the nodal field values without any external constraint. The imposition of a specific excitation represented by the incident plane wave has to be taken into consideration through a boundary constraint that establishes a relationship between the incident field, the boundary nodes, and the interior nodes. In Sections 3 and 4, the surface integral equation will be developed for TM and TE polarization, respectively, and used as a boundary constraint in lieu of the last term in Eq. (3), which cannot be implemented directly since the field on \( \Gamma_H^I \) and \( \Gamma_H^{1/2} \) is not known.

3. SURFACE INTEGRAL EQUATION FOR TM POLARIZATION

The surface integral equation using the free-space Green's function will be used to express the nodal field values on \( \Gamma_o \) in terms of the nodal field values on \( \Gamma_B \) in regions I and II, respectively.

A. Upper Half-Space (Region I)

Let us consider the domain \( \Omega_o^I \) representing the half-space above the PEC (see Fig. 3). In \( \Omega_o^I \) and for the TM polarization case, the electric field vector has only a z-component satisfying Helmholtz's equation,

\[
\nabla^2 E_z(p) + k_0^2 E_z(p) = j \omega \mu J_z(p), \quad p \in \Omega_o^I,
\]

(9)

where \( J_z(p) \) is an electric current source inside \( \Omega_o^I \). Let us introduce the Green's function \( G_z^I(p, p') \), which is the solution due to an electric current filament located at \( p' \) and governed by Helmholtz's equation,

\[
\nabla^2 G_z^I(p, p') + k_0^2 G_z^I(p, p') = - \delta(p - p'), \quad p, p' \in \Omega_o^I,
\]

(10)

and satisfying the boundary condition \( G_z^I(p, p')|_{p=0} = 0 \) (i.e., \( G_z^I = 0 \) on \( \Gamma^I \)) and the Sommerfeld radiation condition at infinity. \( G_z^I(p, p') \) is easily found to be the zeroth-order Hankel function of the second kind given as

\[
G_z^I(p, p') = -j \frac{H_0^1(k_0|p - p'|_{\text{source}})}{4} + \frac{j}{4} H_0^1(k_0|p - p'|_{\text{image source}}).
\]

(11)

Multiplying both sides of Eq. (9) by \( G_z^I(p, p') \) and integrating over \( \Omega_o^I \) yields

\[
\int_{\Omega_o^I} G_z^I(p, p') (\nabla^2 E_z(p) + k_0^2 E_z(p)) d\Omega^I = j \omega \mu \int_{\Omega_o^I} J_z(p) G_z^I(p, p') d\Omega^I.
\]

(12)

invoking Green's second identity

Fig. 3. (Color online) Schematic of the surface integral contour in the upper half-space and lower half-space of the hole.
\[ \int \left( E_z \nabla^2 G^i_x - G_x \nabla^2 E_z \right) d\Omega^f = \int_{1^f \times 1^l_x} \left( \frac{\partial G^i_x}{\partial n} - G^i_x \frac{\partial E_z}{\partial n} \right) d\Gamma^l, \]  
(13)

where \( \Gamma^l + \Gamma^l_x \) is the contour enclosing \( \Omega^l_x \), Eq. (12) can be written as

\[ \int_{\Gamma^l_x} E_z(p)(\nabla^2 G^i_x + k_0^2 G^i_x) d\Omega^f = j\omega \mu \int_{\Gamma^l_x} J_z(p) G^i_x d\Omega^f \]

\[ + \int_{1^f \times 1^l_x} \left( \frac{\partial G^i_x(p)}{\partial n} - G^i_x(p) \frac{\partial E_z(p)}{\partial n} \right) d\Gamma^l. \]  
(14)

Substituting Eq. (10) in Eq. (14), we have

\[ E_z(p') = -j\omega \mu \int_{\Gamma^l_x} J_z(p) G^i_x(p, p') d\Omega^f \]

\[ - \int_{1^f \times 1^l_x} E_z(p') \frac{\partial G^i_x(p, p')}{\partial n} d\Gamma. \]  
(15)

Since both \( E_z \) and \( G^i_x \) satisfy the Sommerfeld radiation condition at infinity, integration over \( \Gamma^l \) (see Fig. 3) on the right-hand side of Eq. (15) vanishes. Notice that \( G^i_x \) is zero on \( \Gamma^l \) [see Eq. (11)]. Additionally, \( E_z(p') \) is zero over the PEC ground plane except at the aperture of the hole. Intercalating primed and unprimed coordinates, Eq. (15) can be simplified to

\[ E_z(p) = -j\omega \mu \int_{\Gamma^l_x} J_z(p) G^i_x(p, p') d\Omega^f \]

\[ - \int_{\text{Aperture}} E_z(p') \frac{\partial G^i_x(p, p')}{\partial n} d\Gamma^l. \]  
(16)

In Eq. (16), the first term of the right-hand side represents the electric field generated by the current filament and its image in the vicinity of the PEC ground plane. The second term in Eq. (16) represents the field perturbation due to the aperture of the hole. In other words, the total electric field at each point in the upper half-space is the sum of the incident field, reflected field due to the PEC surface and the scattered field due to the aperture of the hole as

\[ E_z(p) = E_z^{inc}(p) + E_z^{sc}(p) - \int_{\text{Aperture}} E_z(p') \frac{\partial G^i_x(p, p')}{\partial n} d\Gamma^l. \]  
(17)

Referring to Fig. 2, let \( \rho \) and \( \rho' \) be designated the position of nodes on \( \Gamma^l_x \) and \( \Gamma^l_{x'} \), respectively. Therefore the incident and the reflected waves can be written as

\[ E_z^{inc} = \exp(jk_0(x \sin \theta - y \cos \theta)), \]

\[ E_z^{sc} = -\exp(jk_0(x \sin \theta + y \cos \theta)), \]  
(18)

where \( x \) and \( y \) are Cartesian components of \( \rho \). To calculate the last term in Eq. (17), the aperture, \( \Gamma^l_{x'} \), is discretized into \( n \) segments with length of \( \Delta x' \). We then expand \( E_z(p') \) over \( \Gamma^l_{x'} \) in terms of piecewise linear interpolating functions as

\[ E_z(p') = \sum_{j=1}^{n} E_j \sum_{k=1}^{2} \psi_{jk}(x'_j), \]  
(19)

where \( x' \) and \( y' \) are Cartesian components of \( \rho' \) and

\[ \psi_{jk}(x') = \begin{cases} \frac{x'}{\Delta x'}, & k = 1; \\ 1 - \frac{x'}{\Delta x'}, & k = 2. \end{cases} \]  
(20)

Equation (17) can be represented in matrix notation as

\[ [u_{x'}] = [T^d] + [S^l][u_\rho], \]  
(21)

where the elements of \([u_{x'}], [u_\rho], \) and \([T^d] \) matrices represent \( E_z(p), E_z(p'), \) and \( G^i_x(p, p'), \) respectively, at each node. Noting that \( n' = -y' \), the elements of \([S^l] \) are defined as

\[ S^l_{ij} = \int_{x'_j - \Delta x'}^{x'_j + \Delta x'} \psi_{ij}(x'_j) \frac{\partial G^i_x(x_i, y, x'_j, y')}{\partial y'} \left| _{y'=0} \right. \]  

\[ + \int_{x'_j}^{x'_j + \Delta x'} \psi_{ij}(x'_j) \frac{\partial G^i_x(x_i, y, x'_j, y')}{\partial y'} \left| _{y'=0} \right. \]  
(22)

where

\[ \frac{\partial G^i_x(x_i, y, x'_j, y')}{\partial y'} \left| _{y'=0} \right. = \frac{-jk_0 y}{2\sqrt{(x_i - x'_j)^2 + y'^2}} H^l_0(k_0 \sqrt{(x_i - x'_j)^2 + y'^2}). \]  
(23)

Equation (21) represents the boundary condition on the opening of the hole.

**B. Lower Half-Space (Region II)**

Let us consider the domain \( \Omega^H \) representing the half-space below the PEC slab (see Fig. 3). In the source-free region \( \Omega^H_x \) and for the TM polarization case, the transmitted electric field vector has only a z-component satisfying the homogeneous Helmholtz’s equation:

\[ \nabla^2 E_z(p) + k_0^2 E_z(p) = 0, \quad \rho \in \Omega^H_x. \]  
(24)

Let us introduce the Green’s function \( G^H_x(p, p') \) governed by Helmholtz’s equation:

\[ \nabla^2 G^H_x(p, p') + k_0^2 G^H_x(p, p') = -\delta(p - p'), \quad \rho, \rho' \in \Omega^H_x. \]  
(25)

Equation (25) is similar to Eq. (10), but the main difference is in the location of the delta source. In Eq. (10) we assumed that the delta source is located on \( \Gamma^l_x \), whereas in Eq. (25) the delta source is assumed to be located on \( \Gamma^l_{x'} \). \( G^H_x(p, p') \) satisfies the boundary condition \( G^H_x(p, p') \left| _{y=t} = 0 \right. \) where \( t \) is the thickness of the PEC slab (i.e., \( G^H_x = 0 \) on \( \Gamma^l_{x'} \)) and the Sommerfeld radiation condition at infinity. \( G^H_x(p, p') \) is found to be the zeroth-order Hankel function of the second kind as Eq. (11). Multiplying both sides of Eq. (24) by \( G^H_x(p, p') \) and following a procedure similar to
that of region $I$, we obtain
\[ E_z(p) = -\int_{A_\Omega} E_z(p') \frac{\partial^2 G_z^R(p,p')}{\partial n'} d\Gamma^R, \]  
(26)
where $\Gamma^R + \Gamma^H$ is the contour enclosing $\Omega_2^H$ in the counterclockwise direction. Since both $E_z$ and $G_z^R$ satisfy the Sommerfeld radiation condition at infinity, integration over $\Gamma^R$ (see Fig. 3) on the right-hand side of Eq. (26) vanishes.

Notice that $G_z^R$ is zero on $\Gamma^H$. Additionally, $E_z(p')$ is zero over the PEC ground plane except on the aperture of the hole. On interchanging primed and unprimed coordinates, Eq. (26) can be simplified to
\[ E_z(p) = -\int_{\text{Aperture}} E_z(p') \frac{\partial^2 G_z^R(p,p')}{\partial n'} d\Gamma^H. \]  
(27)
Equation (27) represents the transmitted field from the aperture of the hole into the region $H$. Upon using $x'$ and $y'$ as Cartesian components of $p'$, Eq. (27) can be written as
\[ E_z(p) = -\int_{\text{Aperture}} E_z(p') \frac{\partial^2 G_z^R(p,p')}{\partial y'} dx'. \]  
(28)
It is noticeable that the normal vector $n'$ is in the $y'$ direction. To calculate Eq. (28), the aperture, $\Gamma^H$, is discretized into $n$ segments with length of $\Delta x'$ in the same manner as for the upper half-space. Also we expand $E_z(p')$ over $\Gamma^H$ in terms of piecewise linear interpolating functions as Eq. (19), where the interpolating function is defined as Eq. (20). Equation (28) can be represented in matrix notation as
\[ [u_{i,j}] = -[S^H][u_{i,j}], \]  
(29)
where the elements of $[u_{i,j}]$ represent $E_z(p)$ at each node. The elements of $[S^H]$ are defined as
\[ S^H_{i,j} = \int_{x_j-\Delta x'}^{x_j} \psi_1(x_j') \frac{\partial G_z^R(x_i,y,x_j',y')}{\partial y'} \mid_{y'=t} dx'. \]  
(30)
where
\[ \frac{\partial G_z^R(x_i,y,x_j',y')}{\partial y'} \mid_{y'=t} = -\frac{j k_0(y + t)}{2 \sqrt{(x_i-x_j')^2 + (y + t)^2}}. \]  
(31)
Equation (29) represents the boundary condition on the aperture of the hole in region $H$.

C. Reduced Matrix Form for TM Polarization
Equations (21) and (29) represent the boundary condition on the opening of the hole. Combining these equations and Eq. (8) in matrix form results in the reduced system matrix
\[
\begin{bmatrix}
M_{ii} & M_{i\beta}^T & M_{i\beta}^H \\
M_{\beta i} & M_{\beta\beta}^T & M_{\beta\beta}^H \\
M_{\beta i} & M_{\beta\beta}^T & M_{\beta\beta}^H
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_{i\beta} \\
u_{i\beta}^H
\end{bmatrix}
= \begin{bmatrix}
F_i \\
F_{i\beta} \\
F_{i\beta}^H
\end{bmatrix}.
\]  
(32)

Equation (32) represents the modified system matrix that can be solved using commonly used methods for solving linear systems such as conjugate gradient method (CGM).

4. SURFACE INTEGRAL EQUATION FOR TE POLARIZATION

By modifying the Green’s function, the surface integral equation is derived for the TE polarization in region $I$ and region $H$.

A. Upper Half-Space (Region $I$)
The surface integral equation for the TE polarization case is defined by replacing the electric current filament with a magnetic current filament $M_z$. In $\Omega_0$ and for the TE polarization case, the magnetic field vector has only a $z$-component satisfying the Helmholtz equation,
\[ \nabla^2 H_z(p) + k_0^2 H_z(p) = j \omega M_z(p), \quad \rho \in \Omega_0, \]  
(33)
where $\rho$ and $\Omega_0$ have the same definition as in Section 3 (see Fig. 3). Let us introduce the Green’s function $G_0^T(p,p')$, which is the solution due to the magnetic current filament located at $p'$ and governed by the Helmholtz equation,
\[ \nabla^2 G_0^T(p,p') + k_0^2 G_0^T(p,p') = -\delta(p - p'), \quad p, p' \in \Omega_0. \]  
(34)

Since the image of the magnetic current in the vicinity of the PEC surface is in the same direction as the original current, therefore $G_0^T(p,p')|_{y=0} \neq 0$ on the ground plane. In this case, the boundary condition on the PEC surface is represented as
\[ \frac{\partial G_0^T(p,p')}{\partial y} \mid_{y=0} = 0. \]  
(35)

In addition, $G_0^T(p,p')$ satisfies the Sommerfeld radiation condition at infinity. Therefore $G_0^T(p,p')$ can be represented in terms of the zeroth-order Hankel function of the second kind as
\[ G_0^T(p,p') = -\frac{j}{4} H_0^2(k_0|\rho - p'|_{\text{source}}) - \frac{j}{4} H_0^2(k_0|\rho - p'|_{\text{image source}}). \]  
(36)

Multiplying both sides of Eq. (33) by $G_0^T(p,p')$ and following a procedure similar to that of the TM polarization case, we obtain
Both $H_z(\rho)$ and $G_h(\rho, \rho')$ satisfy the Sommerfeld radiation condition at infinity; therefore integration over $\Omega^I_e$ on the right-hand side of Eq. (37) vanishes. Notice that $\partial H_z(\rho)/\partial n$ is zero over the PEC ground plane except on the aperture of the hole and $\partial G_h(\rho, \rho')/\partial n$ is zero on the $\Gamma^I_e$ [see Eq. (35)]. On interchanging primed and unprimed coordinates, Eq. (37) reduces to

$$H_z(\rho) = -j\omega\int_{\Omega^I_e} M_z(\rho) G_h(\rho, \rho') \, d\Omega^I_e$$

$$- \oint_{\Gamma^I_e} \left( H_z(\rho) \frac{\partial G_h(\rho, \rho')}{\partial n} - G_h(\rho, \rho') \frac{\partial H_z(\rho)}{\partial n} \right) \, d\Gamma^I_e.$$  

(37)

By the same definition of $\rho$ and $\rho'$ and assuming the coordinate system as in Fig. 2, the incident and reflected waves can be written as

$$H^inc_z = \exp(jk_0 y \sin \theta - y \cos \theta),$$

$$H^ref_z = \exp(jk_0 y \sin \theta + y \cos \theta).$$

(40)

To calculate the last term in Eq. (39), the partial derivative $\partial H_z(\rho')/\partial n'$ can be conveniently expressed as a first-order finite difference as

$$\frac{\partial H_z(\rho')}{\partial n'} = - \frac{H_z(x, y') - H_z(x', y')}{y - y'},$$

(41)

(notice that the minus sign on the right-hand side of Eq. (41) is due to $n' = y'$; then the aperture $\Gamma^I_h$ and $\Gamma^I_0$ are discretized into $n$ segments with length of $\Delta y'$. By expanding both $H_z(y)$ and $H_z(y')$ over the aperture of the hole in terms of pulse functions as

$$H_z = \sum_{j=1}^{n} H_{zj} \psi(x'_j),$$

(42)

where

$$\psi(x'_j) = \begin{cases} 1, & x'_j - \frac{\Delta x'_j}{2} < x'_j < x'_j + \frac{\Delta x'_j}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

(43)

the behavior of the fields at the edges of the hole can be captured accurately. By replacing the field expansions in Eq. (39) and defining the matrix $[S^I]$ as

$$S^I_{ij} = \int_{\Gamma^I_e} \frac{x'_i x'_j}{1 + \Delta x'_j/2} g_h(x_i, y, x'_j, y') \psi(x'_j) \, dx',$$

(44)

where

$$g_h(x_i, y, x'_j, y') = \frac{-j}{2} H^0_k(k_0 \sqrt{(x_i - x'_j)^2 + y^2}),$$

(45)

Eq. (39) can be represented in the matrix form as

$$[u^I] = [T^I] - [S^I] [u^I] - [u^I],$$

(46)

where the elements of $[u^I]$, $[u^I]$, and $[T^I]$ matrices represent $H_z(x, y)$, $H_z(x', y')$, and $H^inc_z(x, y) + H^ref_z(x, y)$, respectively, at each node. Equation (46) can be rearranged as

$$[u^I] = [\{1 + [S^I]^\dagger [T^I]\}] [\{1 + [S^I]^\dagger [S^I]\}] [u^I].$$

(47)

Equation (47) represents the boundary condition on the opening of the hole into region $I$.

B. Lower Half-Space (Region $II$)

Let us consider the domain $\Omega^I_h$ representing the source-free half-space below the PEC slab (see Fig. 3). In the source-free region $\Omega^II$ and for the TE polarization case, the transmitted magnetic field vector has only a z-component satisfying the homogenous the Helmholtz’s equation:

$$\nabla^2 H_z(\rho) + k_0^2 H_z(\rho) = 0, \quad \rho \in \Omega^II_h$$

(48)

Let us introduce the Green’s function $G^H_h(\rho, \rho')$ governed by Helmholtz’s equation:

$$\nabla^2 G^H_h(\rho, \rho') + k_0^2 G^H_h(\rho, \rho') = -\delta(\rho - \rho') \quad \rho, \rho' \in \Omega^II_h.$$  

(49)

Notice that Eq. (49) differs from Eq. (34) in the location of the delta source. In Eq. (34) we assumed that delta source is located on $\Gamma^I_0$, whereas in Eq. (49) the delta source is assumed to be located on $\Gamma^II_0$. $G^H_h(\rho, \rho')$ satisfies the Sommerfeld radiation condition at infinity and the boundary condition as

$$\frac{\partial G^H_h(\rho, \rho')}{\partial n} \bigg|_{y=t} = 0,$$

(50)

where $t$ is the thickness of the PEC slab. $G^H_h(\rho, \rho')$ is found to be the zeroth-order Hankel function of the second kind as Eq. (36). Multiplying both sides of Eq. (48) by $G^H_h(\rho, \rho')$ and following a procedure similar to that of region $I$, we obtain
By replacing the field expansions in Eq. (53) and de-aperture of the hole in terms of pulse functions as Eq. (52). Sommerfeld radiation condition at infinity, integration over \( \Gamma^H \) (see Fig. 3) on the right-hand side of Eq. (51) vanishes. Notice that \( \partial G^H_n(\rho, \rho')/\partial n \) is zero on \( \Gamma^H \). Additionally, \( \partial H_z(\rho)/\partial n \) is zero over the PEC ground plane except on aperture of the hole. On interchanging primed and unprimed coordinates, the Eq. (51) can be simplified to

\[
H_z(\rho) = \int_{\text{Aperture}^H} G^H_n(\rho, \rho') \frac{\partial H_z(\rho')}{\partial n'} d\Gamma^H, \tag{52}
\]

Equation (52), represents the transmitted field from the aperture of the hole into the region \( II \). Upon using \((x', y')\) and \((x, y)\) as a Cartesian components of \( \rho' \) and \( \rho \) respectively, and expressing the partial derivative as a first-order finite difference, Eq. (52) can be written as

\[
H_z(\rho) = \int_{\text{Aperture}^H} G^H_n(\rho, \rho') \frac{H_z(x = x', y) - H_z(x', y')}{y - y'} dx'. \tag{53}
\]

To calculate the integral in Eq. (53), the aperture \( \Gamma^H \), and \( \Gamma^0_B \) are discretized into \( n \) segments with length of \( \Delta x' \). Both \( H_z(x, y) \) and \( H_z(x', y') \) are expanded as Eq. (42) over the aperture of the hole in terms of pulse functions as Eq. (43). By replacing the field expansions in Eq. (53) and defining of the matrix \( [S^H] \) as

\[
[S^H] = \int_{x' - \Delta x'/2}^{x' + \Delta x'/2} G^H_n(x_i, y, x'_j, y' = -t) \psi_j(x'_j) y - y' dx', \tag{54}
\]

where

\[
G^H_n(x_i, y, x'_j, y') = \frac{-j}{2} H_0^2(k_0 \sqrt{(x_i - x'_j)^2 + (y + t)^2}, \tag{55}
\]

Eq. (53) can be represented in the matrix form as

\[
[u_{\mu}] = [S^H] [u_{\nu}] - [u_{\nu}], \tag{56}
\]

where the elements of \([u_{\mu}] \) and \([u_{\nu}] \) matrices represent \( H_z(x, y) \) and \( H_z(x', y') \), respectively, at each node. Equation (56) can be rearranged as

\[
[u_{\mu}] = -([1] - [S^H]^{-1} [S^H]) [u_{\nu}], \tag{57}
\]

Equation (57) represents the boundary condition on the opening of the hole into region \( II \).

**C. Reduced Matrix Form for TE Polarization**

Equations (47) and (57) represent the boundary condition on the openings of the hole. Combining these equations and Eq. (8) in matrix form results in the reduced system matrix as

\[
\begin{bmatrix}
M_{\mu I} & M_{\mu II} & M_{\mu B} \\
M_{\nu I} & M_{\nu II} & M_{\nu B} \\
M_{\nu II} & M_{\nu II} & M_{\nu II}
\end{bmatrix}
\begin{bmatrix}
[u_{\mu}] \ 
[u_{\nu}] \\
F_{\mu} \\
F_{\nu}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \tag{58}
\]

5. EXTENSION TO MULTIPLE HOLES

In this section, we extend the method developed above to the problem of scattering from multiple holes in a PEC surface. As an example, a schematic representing the scattering problem for two holes is shown in Fig. 4. The regions inside the holes are labeled \( \Omega_1 \) and \( \Omega_2 \). Extending the finite-element development in Section 2 to two holes, we generalize the system matrix for the domains \( \Omega_1 \) and \( \Omega_2 \) as

\[
[M]^{1}[u]^{1} = [F]^{1}, \tag{59}
\]

\[
[M]^{2}[u]^{2} = [F]^{2}, \tag{60}
\]

where each system of equations can be represented symbolically as Eq. (8). Assembling the two systems using global numbering of nodes gives

\[
\begin{bmatrix}
[M]^{1} & 0 \\
0 & [M]^{2}
\end{bmatrix}
\begin{bmatrix}
[u]^{1} \\
[u]^{2}
\end{bmatrix}
= \begin{bmatrix}
[F]^{1} \\
[F]^{2}
\end{bmatrix}. \tag{60}
\]

The two system matrices arising from each of the two holes will be coupled through the surface integral equation in the following manner. In region \( I \), each node on \( \Gamma^0_B \) is connected to the Green’s function to all the nodes on the aperture of the two holes, \( \Gamma^0_B \), via Eq. (17) and Eq. (39) for TM and TE polarization, respectively (see Fig. 4). Also, in region \( II \), each node on \( \Gamma^0_B \) is connected via the Green’s function to all the nodes on the aperture of the two holes, \( \Gamma^0_B \), via Eq. (27) and Eq. (52) for the TM and TE polarization, respectively. In other words, the holes are coupled to each other only through the surface integral equation and the Green’s function in each region. In (17) and (39), or in (27) and (52), the integration is performed over the apertures of both holes in region \( I \) or region \( II \), respectively. For instance, Eq. (21) and Eq. (29) for the TM polarization can be represented symbolically in matrix form as

![Fig. 4.](image-url) (Color online) Schematic showing the extension of the surface integral method to multiple holes with side cavities.
\[
\begin{bmatrix}
[u_p^{(1)}] \\
[u_p^{(2)}]
\end{bmatrix} = \begin{bmatrix}
[T^{(1)}] \\
[T^{(2)}]
\end{bmatrix} + \begin{bmatrix}
[S^{(1)}]^{(1)} & [S^{(1)}]^{(12)} \\
[S^{(2)}]^{(2)} & [S^{(2)}]^{(12)}
\end{bmatrix} \begin{bmatrix}
[u_h^{(1)}] \\
[u_h^{(2)}]
\end{bmatrix}
\] (61)

and
\[
\begin{bmatrix}
[u_s^{(1)}] \\
[u_s^{(2)}]
\end{bmatrix} = \begin{bmatrix}
[S^{(1)}]^{(1)} & [S^{(1)}]^{(12)} \\
[S^{(2)}]^{(2)} & [S^{(2)}]^{(12)}
\end{bmatrix} \begin{bmatrix}
[u_h^{(1)}] \\
[u_h^{(2)}]
\end{bmatrix},
\] (62)

respectively. \([S]^{(j)}\) represents the connectivity between nodes on \(\Gamma_O\) of the \(i\)th hole \([u_h^{(i)}]\) and nodes on \(\Gamma_B\) of the \(j\)th hole \([u_s^{(j)}]\) via the surface integral equation \((i,j=1,2)\) in both region I and II. Combining Eq. (60), Eq. (61) and Eq. (62) in matrix form results in the reduced system matrix as

\[
\begin{bmatrix}
[M']^{(1)} \\
[M']^{(2)}
\end{bmatrix} = \begin{bmatrix}
[M_{i,i'}^{(1)}] & [M_{i,i'}^{(12)}] & [M_{i,j'}^{(1)}] \\
[M_{j,j'}^{(2)}] & [M_{j,j'}^{(22)}] & [M_{j,j'}^{(2)}]
\end{bmatrix},
\] (63)

where \([C]^{(12)}\) and \([C]^{(21)}\) are matrices representing the coupling between the two holes and are given by
\[
[C]^{(12)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & [M_{i,i'}^{(1)}][S^{(1)}]^{(12)} & 0 \\
0 & 0 & -[M_{j,j'}^{(2)}][S^{(2)}]^{(12)}
\end{bmatrix},
\] (64)

and \([M']\) and \([F']\) are given by
\[
[M']^{(1)} = \begin{bmatrix}
[M_{i,i'}^{(1)}] \\
[M_{j,j'}^{(2)}]
\end{bmatrix},
\] (65)

and \([u']^{(k)}\) is given by
\[
[u']^{(k)} = \begin{bmatrix}
[u_i^{(k)}] \\
[u_i^{(k)}] \\
[u_s^{(k)}]
\end{bmatrix} (k = 1 & 2).
\] (66)

Notice that the coupling matrices \([C]^{(12)}\) and \([C]^{(21)}\) are not necessarily identical. A similar procedure is applicable to TE polarization. Generalizing the formulation to \(N\) holes results in the following system matrix:
\[
\begin{bmatrix}
[M']^{(1)} & [C]^{(12)} & \cdots & [C]^{(1N)} \\
[C]^{(21)} & [M']^{(2)} & \cdots & [C]^{(2N)} \\
\vdots & \vdots & \ddots & \vdots \\
[C]^{(N1)} & [C]^{(N2)} & \cdots & [M']^{(N)}
\end{bmatrix} \begin{bmatrix}
[u']^{(1)} \\
[u']^{(2)} \\
\vdots \\
[u']^{(N)}
\end{bmatrix} = \begin{bmatrix}
[F']^{(1)} \\
[F']^{(2)} \\
\vdots \\
[F']^{(N)}
\end{bmatrix},
\] (67)

where
\[
[C]^{(pq)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & [M_{i,i'}^{(p)}][S^{(1)}]^{(pq)} & 0 \\
0 & 0 & -[M_{j,j'}^{(q)}][S^{(2)}]^{(pq)}
\end{bmatrix},
\]
The formulation presented in this work is applicable to holes with side cavities present in perfectly conducting surfaces. For multiple holes and cavities, Eq. (67) gives a mathematical quantification of the coupling factors between the holes and cavities. Physically, we expect the holes and cavities to be coupled through surface currents existing on the segments connecting the holes and cavities, as that is the only mechanism for energy transfer between them.

6. NUMERICAL RESULTS

Once the system of linear equations, Eq. (32) for the TM polarization or Eq. (58) for the TE polarization, is derived, its solution can be obtained using commonly used methods for solving linear systems. In this work, we use the conjugate gradient method (CGM) to solve the linear system of equations. In this section, we provide examples of different holes and provide comparison with results obtained using the commercial finite-element simulator HFSS [12]. Throughout this work, the solution obtained using the method presented in this paper is referred to as FEM-TFSIE. Without loss of generality, the magnitude of the incident electric field is assumed to be unity. To discretize the solution domain, we use first-order triangle elements with mesh density of approximately 20 nodes per \( \lambda \) for the TM case. To accurately capture the behavior of the field at the edges of the holes in the TE case, we use mesh density of 100 nodes per \( \lambda \). Notice that in such polarization, the H field on the PEC surface corresponds to surface current flowing in directions orthogonal to the H field.

A. Single-Hole Case

In the first example, we considered a \( 0.8\times 0.5\lambda \) (width \times depth) rectangular hole, where \( \lambda \) is the wavelength in free space. Figure 5 shows the total electric field at both apertures of the hole for TM incident plane wave for an oblique incident angle. The results in Fig. 5 show strong agreement between the calculations using FEM-TFSIE and those obtained using HFSS. In the case of HFSS, where an absorbing boundary condition is used, the required computational domain needed to achieve similar accuracy was approximately 18\( \lambda \)^2. Notice that the FEM-TFSIE solution space was confined only to the area of the hole of 0.4\( \lambda \)^2.

Figure 6 shows the total magnetic field at both apertures of the same hole, for the case of TE incident plane wave. We observe that results from our method are in good agreement with those obtained by HFSS despite a small deviation that is pronounced at the edges of the hole. The reason for this deviation is that we used triangle elements of a constant size since our meshing scheme is performed without any automated mesh generator. If an adaptive mesh scheme is employed as was the case in HFSS, the deviation in the field at the corner location can be eliminated.
As the next example of a single-hole case, we consider a $0.7\times 0.35$ rectangular hole shown in Fig. 7. The hole is filled with silicon having a relative permittivity of $\varepsilon_r = 11.9$ and including a $0.42\times 0.07$ PEC strip positioned at the geometric center of the hole. The electric field at the aperture for the TM case is calculated. This numerical example shows the versatility of FEM-TFSIE for solving the problem of holes with complex structures and fillings.

B. Multiple Holes

We consider five identical $0.4\times 0.2$ rectangular holes separated by $0.4\lambda$. Figure 8 shows the total electric field at the apertures of the five holes for oblique incident plane waves with TM polarization. Close agreement between FEM-TFSIE and HFSS is observed. Figure 9 shows the total magnetic field at the aperture of the same structure for the TE case, for oblique incidence. We observe close agreement between FEM-TFSIE and HFSS. In Figs. 10 and 11 we present the far-field in region II for the array of five holes. The far field is calculated from the aperture fields using the surface integral equation. Figure 10 shows the far field for TM-polarized normal incidence and Fig. 11 shows the far field for TE-polarized oblique incidence.

C. Single Hole with Side Grating

Here we consider a single $0.5\times 0.8$ rectangular hole and three $0.5\times 0.3$ cavities as a side grating in the PEC slab (see inset of Fig. 12). The hole and cavities are separated by $0.5\lambda$. The two sides of the PEC slab have identical gratings. The analysis of this structure is important and highly relevant in the study of the phenomenon of extraordinary transmission of light.

Figure 12 shows the total electric field at the apertures of the hole and side cavities for oblique incident plane waves with TM polarization. Close agreement between FEM-TFSIE and HFSS is observed. However in the case...
of HFSS, the required computational domain needed to achieve the shown accuracy was approximately $32.2\lambda^2$, while the FEM-TFSIE solution space was confined to the hole and cavities area of $2.2\lambda^2$. This significant enhancement in efficiency facilitates further studies of plasmonic resonance due to grating surfaces. In Figs. 13 and 14, we show the total magnetic field at the apertures for the TE case, for normal and oblique incidence. We observe a deviation between FEM-TFSIE and HFSS. Each cavity in region II is coupled to the hole and the other cavities via surface current excited on the PEC segments. The surface current is excited by the incident field in region I. The surface current on the surface in region II supports propagating waves that graze the upper and lower boundaries of the computational domain when using ABC or PML to truncate the computational domain. To accurately account for such waves, the computational domain when using the HFSS solver has to be enlarged appreciably. In fact, we show that by increasing the distance between the computational domain boundaries and the aperture of the hole and cavities from $h=1\lambda$ in Fig. 14 to $h=3\lambda$ in Fig. 15, the deviation between the results obtained by HFSS and FEM-TFSIE decreases significantly.

7. CONCLUSION

In this study, the finite-element method and the surface integral equation were combined to solve the problem of scattering from single and multiple holes and a single hole with side grating. The surface integral equation employing Green's function was used to truncate the mesh region. In this formulation no singularities in Green's function arises while applying the surface integral as a boundary constraint. The formulation is based on the to-
tal field and overcomes the limitations of earlier methods based on the surface integral equation, and it is applicable to both TM and TE polarizations.

Several numerical examples were presented for single and multiple holes. The solutions using FEM-TFSIE were in close agreement with results obtained by a commercial finite-element simulator. Furthermore, the FEM-TFSIE is very versatile in handling complex structures of holes with inhomogeneous fillings without any modification to the algorithm. The run time and solution efficiency of FEM-TFSIE are two major attractive features of this method, making it well suited for optimization problems involving scattering from gratings in metallic screens.

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