On Non-Binary Constellations for Channel-Coded Physical-Layer Network Coding

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***Abstract*—We investigate channel-coded physical-layer network coding in a two-way relaying scenario, where the end nodes A and B choose their symbols, $\mathbf{S}_A$ and $\mathbf{S}_B$, from a small non-binary field, $\mathbb{F}$, and adopt a non-binary PSK modulation. The relay then directly decodes the network-coded combination $a\mathbf{S}_A + b\mathbf{S}_B$ over $\mathbb{F}$ from the noisy received superimposed channel-encoded packets. The advantage of working over non-binary fields is that it offers the opportunity to decode according to multiple decoding coefficients (a, b). As only one of the network-coded combinations needs to be successfully decoded, a key advantage is then a reduction in error probability by attempting to decode against all choices of (a, b). In this paper, we compare different mappings between $\mathbb{F}$ and the PSK constellation, and prove that many have identical performance in terms of frame error rate (FER). Moreover, we derive a lower bound on the performance of decoding the network-coded combinations. Simulation results show that if we adopt either i) concatenated Reed-Solomon and convolutional coding or ii) low-density parity check codes, our non-binary constellations can outperform the binary case significantly in the sense of minimizing the FER and, in particular, the *ternary* constellation has the best FER performance among all considered cases.**

***Index Terms*—Physical-layer network coding, non-binary constellation mappers, outage probability, Reed Solomon Convolutional code concatenation, low density parity check code.**

I. INTRODUCTION

INTERFERENCE, traditionally considered to be destructive to wireless communications, may in fact contain beneficial information. This point of view suggests the use of decoding techniques to process interference in wireless networks, instead of treating it as a nuisance to be avoided [1].

Inspired by the network coding principle [2], [3], physical-layer network coding (PNC) is a technique in which an intermediate node relays a function of the decoded incoming packets, usually linear combinations, rather than the packets individually. In PNC, the combinations are inferred directly from the received signal of the intermediate node. The basic idea of PNC has been proposed independently by several research groups in 2006: Zhang, Liew, and Lam [4], Popovski and Yomo [5], and Nazer and Gastpar [6]. Because of its simplicity and the substantial benefits foreseen in it [7], [8], PNC has gained much attention since 2006. Many strategies have been proposed for PNC, with a particular focus on bidirectional relaying, where nodes A and B exchange information with the help of a relay node R. In [4] and [5], three different protocols for bidirectional relaying are presented. Compared to the 4- and 3-stage protocols, the 2-stage protocol can improve throughput because of its effective time usage. In this paper, we concentrate on the 2-stage relaying scheme consisting of an uplink phase and a downlink phase. In the uplink phase, termed *multiple access (MAC)* stage, nodes A and B transmit packets to the relay node R simultaneously. Relay node R then constructs a network-coded packet based on the overlapped signals received from nodes A and B. In the downlink phase, termed *broadcast (BC)* stage, the relay R broadcasts the packet to nodes A and B. Knowing its own information *a priori*, node A (B) can decode the data from node B (A), using the signal broadcast from node R.

The performance of a two way relaying system in 2-, 3- and 4- stage scenarios is investigated in [9]. The authors, in [9], showed that the two-stage PNC scheme offers a higher maximum sum-rate, but a lower sum-bit error rate (BER), than the 4-stage scheme for a number of practical scenarios. They also showed that the 3-stage scheme offers a good compromise between the 2- and 4- stage schemes, and also achieves the best maximum sum-rate and/or sum-BER in certain practical scenarios.

In [10], the authors investigated the use of structured and lattice codes in a scenario for two-way relaying. In [11], a compute-and-forward strategy is proposed where the relays, knowing the channel coefficients, decode linear functions of transmitted messages. The authors used lattice codes whose algebraic structure ensures that integer combinations of codewords can be decoded reliably.

In [12], the authors considered the use of non-coherent detection at the relay for a PNC scenario. The proposed non-coherent relay does not require phase synchronism.

Network coding at the relay node R is challenging because of the fact that channel gains and noise at the MAC stage randomly perturb the received overlapped packets. In [13], the authors introduce a modulation design method for dealing with this randomness which improves the throughput significantly. Their scheme employs the use of unusual 5-ary modulation in the BC stage while QPSK modulation is used in the MAC stage. In their model, a denoise-and-forward (DF) scheme is implemented at the relay node.

In [14], the overlapped BPSK-modulated signals in the relay node R are transformed directly to the network-coded packets.

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The authors adopt a repeat accumulate (RA) channel code at the two end nodes and redesign the belief propagation decoding algorithm of the RA code to suit the PNC configuration.

In [15], the authors presented a new non-uniform high-order M-PAM constellation design that allows for a computationally efficient binary interleaved coded modulation for PNC over a DF relaying channel. By comparison, in [16], the authors considered a memoryless bidirectional relaying system where the signal transmitted by a relay depends only on its last received signal, i.e., the system is uncoded. For binary antipodal signaling, they considered so called absoluted (abs) based schemes. Moreover, they analyzed existing and new relay strategies as well as optimized abs and non-abs based schemes via functional analysis to minimize the average probability of error over all possible relay functions.

In [17], the authors have introduced two new PNC categories: PNCF (PNC over finite field) and PNCI (PNC over infinite field) according to whether the network-code field adopted is finite or infinite. In their model, the source packets are not channel-coded. However, the idea of adopting channel-coding schemes at the end nodes has been investigated in [18], [19]. Two comprehensive surveys about PNC can be found in [20], [21].

In this paper, the end nodes A and B choose their symbols, $\mathbf{S}_A$ and $\mathbf{S}_B$, from a finite field $\mathbb{F} = GF(q)$. The symbols are then q-PSK modulated $(q = 2, 3, 4, 5)$ and sent to the relay. We propose a PNCF scheme for directly decoding a network-coded combination, i.e., $a\mathbf{S}_A + b\mathbf{S}_B$ over $\mathbb{F}$, from the overlapped channel-coded signals received from the two end nodes (plus noise). This can be seen as a compute-and-forward type approach since a linear function of the transmitted messages is inferred from the noisy linear combination provided by the channels. However, as opposed to lattice codes, the linear coding schemes here are either a practical concatenation of Reed-Solomon and convolutional codes (RS-CC) or a low density parity check code (LDPC), over a small non-binary field, and are suitable for both quasi-static and fast fading channels. To the best of our knowledge, practical physical-layer network coding for fast fading channels has not been studied in the literature. Fast fading channels are a good model for frequency selective channels where orthogonal frequency division multiplexing (OFDM) is applied with subcarrier interleaving. Furthermore, due to the cyclic prefix of OFDM, if the two transmissions from the end nodes are not perfectly received in time at the relay, the timing error can be absorbed as part of the channel responses.

For the binary case, the only possible network-coded combination, $\mathbf{S}_A + \mathbf{S}_B$ over the binary field, does not offer the best performance in several channel conditions [13]. Since unconventional non-binary constellations offer flexibility in the choice of decoding coefficients (a, b), the relay is capable of attempting to decode multiple network-coded combinations. If at least one of these network-coded combinations is decoded successfully, a correct decision will be made at the relay, i.e., attempting to decode against all combinations (a, b) can decrease the probability of error at the relay. Note that since a cyclic redundancy check (CRC) code over $\mathbb{F}$ is linear, a correctly decoded combination will pass a CRC test.

We aim to compare cases when end nodes use constellations of size $q = 3, 4, 5$ with the conventional binary case. Simulation results suggest that further increasing the constellation size is not beneficial since for a fixed transmission power, constellation points get closer to each other and the probability of error increases. When the end nodes employ channel coding, we find that non-binary constellations can outperform the binary case as decoding against all coefficients (a, b) provides greater benefit than the reduction in minimum distance costs. The major contributions of this paper are summarized as follow:

- We utilize non-binary constellations and directly decode network-coded combinations, $a\mathbf{S}_A + b\mathbf{S}_B$, over finite fields from the superimposed channel-coded packets, using either a practical RS-CC concatenation or LDPC code.
- Working over non-binary finite fields offers multiple choices for decoding coefficients, a and b. We benefit from attempting to decode multiple network-coded combinations at the relay.
- We show that for a finite field $\mathbb{F}$, there are effectively only $|\mathbb{F}| - 1$ pairs of decoding coefficients that should be attempted by the relay. There is no performance gain in attempting more.
- We investigate the performance of different mappings from $\mathbb{F}$ to the $q-$PSK constellation (called constellation mappers, and of which there are $q!$) in terms of FER for $q \leq 5$. We prove that the performance of all constellations mappers in $GF(3)$ is the same in terms of FER for *any* code (linear or non-linear). However, in $GF(2^2)$ the performance of different constellation mappers is the same only if a linear code is adopted at the end nodes. For $GF(5)$, if the code is linear, we show that there are 4 different classes of constellation mappers with possibly different performances, which greatly reduces the search space from $5! = 120$ cases to 4.
- We find a lower bound using Fano's inequality on the FER performance of decoding the network-coded combinations at the relay.
- When using RS-CC, for quasi-static Rayleigh fading channels, finite fields $GF(3)$, $GF(2^2)$ and $GF(5)$ outperform the binary case. For Rayleigh fast fading channels, finite fields $GF(3)$ and $GF(2^2)$ outperform the binary case by attempting to decode multiple network-coded combinations.
- When using LDPC codes, for quasi-static Rayleigh fading and Rayleigh fast fading channels, the finite fields $GF(3)$ and $GF(2^2)$ outperform the binary case by attempting to decode multiple network-coded combinations.
- $GF(3)$ has the best performance in terms of FER among all other fields, for both quasi-static Rayleigh and fast fading Rayleigh channels, when RS-CC or LDPC channel coding is adopted at the MAC stage.

The remainder of this paper is organized as follows: in section II, we introduce the basic system model and investigate the number of network-coded combinations that the relay should attempt to decode. In section III, we explain the system model of channel-coded PNC. In section IV, we investigate the performance of different constellation mappers in terms of FER. In section V, we derive a lower bound on the FER performance of decoding at the relay. In section VI, simulation results are presented. Finally, we conclude the paper in section VII.

II. Bidirectional Relaying

A. Multiple Access (MAC) Stage

Let us assume that each node is equipped with a single antenna and the channel is half duplex. Thus, transmission and reception at a particular node must happen in different time slots. We also assume that there is no direct link between nodes A and B. We denote by $S_A, S_B \in \mathbb{F}$, where $\mathbb{F}$ is a finite field, the source data from A and B, respectively. We let $\mathcal{M} : \mathbb{F} \rightarrow \mathbb{C}$ denote a $q-$PSK constellation mapper used at the MAC stage. The signals transmitted from sources A and B are $X_A = \mathcal{M}(S_A)$ and $X_B = \mathcal{M}(S_B)$. We assume that the constellation points have unit energy. The received signal at the relay node R is expressed as

$$Y_R = H_A X_A + H_B X_B + Z_R, \tag{1}$$

where H_A and H_B are complex-valued channel gains from the end nodes A and B to the relay node R, respectively. We assume that Z_R is complex additive white Gaussian noise (AWGN) with variance σ^2. For a given constellation mapper $\mathcal{M}$ and channel gains H_A and H_B, we will call the following set the *received constellation*:

$$\mathbf{M}_{\mathcal{M}}(H_A, H_B) = \{H_A \mathcal{M}(S_A) + H_B \mathcal{M}(S_B) | S_A, S_B \in \mathbb{F}\}.$$

For simplicity of analysis and exposition, we assume a time-synchronous communication system. Again, this is a reasonable assumption, as an OFDM-based bidirectional relaying system is robust to time synchronization errors due to the cyclic prefix.

B. Physical Layer Network Coding Over Non-Binary Fields

In network coding the critical issue is how the relay R makes use of Y_R to construct a packet to broadcast to the end nodes A and B in the downlink phase. For binary, the most well-known network coding scheme is the XOR or modulo-2 addition:

$$C(S_A, S_B) = S_A \oplus S_B = S_A + S_B \ \ (mod\ 2)\,. \tag{2}$$

As explained in [13], the channel gains do not always favor such a simple XOR code. Hence, decoding multiple network-coded combinations can be helpful, since for successful detection at the relay only one of these combinations needs to be successfully decoded. Working over non-binary fields gives the relay node R the opportunity to map the received signal, Y_R, into multiple network-coded combinations; i.e., $S_R = aS_A + bS_B$ over $\mathbb{F}$, where $a, b \in \mathbb{F}\backslash\{0\}$. Decoding according to a fixed choice (a, b) partitions the q^2 received constellation points, $\mathbf{M}_{\mathcal{M}}(H_A, H_B)$, into q sets according to the level sets of the function $C_{(a,b)}(S_A, S_B) := aS_A + bS_B$ of S_A and S_B. However, not all of these pairs of decoding coefficients lead to a new partitioning of constellation points and thus do not provide a new opportunity for successful decoding at the relay. Those that generate the same partitioning of points either all succeed or all fail when decoding a particular received transmission. The following theorem indicates which of the coefficient pairs, (a, b), are duplicates in this sense.

Theorem 1: The number of pairs of decoding coefficients that provide distinct partitions is $|\mathbb{F}| - 1$. Picking the coefficients $a = 1$ and $b \in \mathbb{F}\backslash\{0\}$ yields all the distinct partitions.

Proof: The proof consists of two parts: 1) any pair of decoding coefficients, (a, b), where $a, b \in \mathbb{F}\backslash\{0\}$ and $a \neq 1$ yields the same partition as the pair $(1, b')$, for some $b' \in \mathbb{F}\backslash\{0\}$ and 2) there are no distinct pairs with an identical first coefficient that yield the same partition.

Part 1: The partition with $(1, b)$, where $b \in \mathbb{F}\backslash\{0\}$, is the same as that with (c, cb), where $c \in \mathbb{F}\backslash\{0\}$, since

$$cC_{(1,b)}(S_A, S_B) = cS_A + cbS_B = C_{(c,cb)}(S_A, S_B), \tag{3}$$

i.e., decoding coefficients $(1, b)$ and (c, cb) produce the same partitioning as they have the same level sets.

Part 2: Assume that the partition with (a, b) is the same as that with (a, b'). Denote by $P \subset \mathbb{F}^2$ a common level set with image $k \in \mathbb{F}$ and $k' \in \mathbb{F}$ for (a, b) and (a, b'), respectively, i.e., for all $(S_A, S_B) \in P$,

$$\begin{aligned} aS_A + bS_B &= k \\ aS_A + b'S_B &= k'. \end{aligned} \tag{4}$$

According to (4), $(b' - b)S_B = k - k'$ for all S_B in $\{S | (S_A, S) \in P\} = \mathbb{F}$. But, this will only hold if $b = b'$ and $k = k'$. Therefore, there are no duplicate pairs with an identical first coefficient. ■

Knowing the channel gains, the relay R performs soft-decision decoding to estimate S_R from the received signal, Y_R, using the probability $\Pr\left(Y_R | aS_A + bS_B = k, H_A, H_B\right)$ for $k \in \mathbb{F}$, computed as follows

$$\Pr\left(Y_R | aS_A + bS_B = k, H_A, H_B\right) = \frac{1}{q\pi\sigma^2} \sum_{\substack{(S_A, S_B) \\ aS_A + bS_B = k}} \exp\left(\frac{-|Y_R - H_A \mathcal{M}\left(S_A\right) - H_B \mathcal{M}(S_B)|^2}{\sigma^2}\right). \tag{5}$$

In the BC stage, the end node A, knowing its own information and the channel gain H_A (obtained say, from pilot symbols), can obtain its intended data, S_B, using the signal broadcast form node R. Similarly, the end node B can extract S_A from S_R.

III. Channel-Coded Bidirectional Relaying

Given that the relay node R attempts to decode against multiple pairs of decoding coefficients, (a, b), in this section, we now explain the channel-coded PNC system model. We let $\mathcal{T} : \mathbb{F}^k \rightarrow \mathbb{F}^n$ denote a linear channel encoder. Denote by $\mathbf{S}_A, \mathbf{S}_B \in \mathbb{F}^k$ the un-encoded data to be transmitted from A and B. The end nodes employ a coding scheme of rate $r = k/n$. The encoded packets are then modulated by $\boldsymbol{\mathcal{M}}$:

$\mathbb{F}^n \rightarrow \mathbb{C}^n$, where the PSK constellation mapper $\mathcal{M} : \mathbb{F} \rightarrow \mathbb{C}$ is applied element by element, as

$$\mathbf{X}_A = \boldsymbol{\mathcal{M}}\left(\mathcal{T}(\mathbf{S}_A)\right)$$
$$\mathbf{X}_B = \boldsymbol{\mathcal{M}}\left(\mathcal{T}(\mathbf{S}_B)\right). \tag{6}$$

Let $\mathbf{U}_A = \mathcal{T}(\mathbf{S}_A)$ and $\mathbf{U}_B = \mathcal{T}(\mathbf{S}_B)$ denote the encoded packets from the end nodes. In the MAC stage, the received signal at the relay node R during the j-th symbol is written as

$$Y_R(j) = H_A(j)X_A(j) + H_B(j)X_B(j) + Z_R(j), \tag{7}$$

where $Z_R(j)$ is complex-valued circularly symmetric AWGN with variance σ^2 and $H_i(j)$ for $i \in \{A, B\}$ is a complex-valued channel gain from the end node i to the relay node R. In this paper, we first consider quasi-static Rayleigh fading channels, where $H_i(j)$ is a constant for all $j \in \mathbb{Z}_n$. When we consider Rayleigh fast fading channels, the $H_i(j)$ are assumed to be i.i.d. for all $j \in \mathbb{Z}_n$.

We aim to directly decode $a\mathbf{S}_A + b\mathbf{S}_B$ over $\mathbb{F}$, using the fact that $\mathcal{T} : \mathbb{F}^k \rightarrow \mathbb{F}^n$ is a linear code. The linearity of the code guarantees

$$\mathcal{T}(a\mathbf{S}_A + b\mathbf{S}_B) = a\mathcal{T}(\mathbf{S}_A) + b\mathcal{T}(\mathbf{S}_B). \tag{8}$$

If from $\mathbf{Y}_R$ we can obtain the soft metrics $\Pr(Y_R(j)|aS_A(j) + bS_b(j) = k, H_A(j), H_B(j))$ for $j \in \mathbb{Z}_n$ and $k \in \mathbb{F}$ and if the linear code allows for soft-input decoding, then $a\mathbf{S}_A + b\mathbf{S}_B$ can be directly decoded without first decoding $\mathbf{S}_A$ and $\mathbf{S}_B$. According to *Theorem* 1, the relay is not required to attempt to decode all network-coded combinations, $a\mathbf{S}_A + b\mathbf{S}_B$, as there are only $|\mathbb{F}| - 1$ effective distinct sets of pairs of decoding coefficients (a, b). In section V and thereafter all channel codes are assumed to be linear.

IV. $q-$PSK Constellation Mappers

Generally, in a field of size q, there are $q!$ different constellation mappers that place the constellation points uniquely on a q-PSK constellation. However, due to symmetric properties of the constellation and the fact that linear codes are employed at the end nodes, the number of constellation mappers that result in different performance in terms of FER, called *distinct mappers* for short, is less. Consider a constellation mapper $\mathcal{M}$ that generates the constellation $\mathbf{c}_1 = (c_{11}, c_{12}, ..., c_{1q})$, where $c_{1i} \in \mathbb{F}$, $c_{1i} \neq c_{1j}$ for $i \neq j$ and $i, j \in \{1, 2, ..., q\}$, with a given FER; without loss of generality, let c_{11} be located at the right corner of the q-PSK constellation, followed by $c_{12}, ..., c_{1q}$ in a counter-clockwise order. The constellation $\mathbf{c}_1$ can be transformed to another constellation called $\mathbf{c}_2 = (c_{21}, c_{22}, ..., c_{2q})$, where $c_{2i} \in \mathbb{F}$, $c_{2i} \neq c_{2j}$ for $i \neq j$ and $i, j \in \{1, 2, ..., q\}$, by applying a permutation such as rotation, or reflection, etc. Let us denote by $\mathcal{P} = \{p_i | 1 \leq i \leq q!\}$ the group of all $q!$ permutations. Without loss of generality assume that the elements of $\mathbb{F}$ are ordered as $\{\xi_1, \xi_2, ...\xi_q\}$. Denote by $\mathcal{C} = \{\mathbf{c}_i | 1 \leq i \leq q!\}$ the set of all constellations created by applying the elements of $\mathcal{P}$ to $\mathbf{c}_I = (\xi_1, \xi_2, ..., \xi_q)$, an *identity constellation*. Since there is a one-to-one correspondence between the elements of $\mathcal{C}$ and those of $\mathcal{P}$, there exists an isomorphism between $\mathcal{P}$ and $\mathcal{C}$ and hence $\mathcal{C}$ can be seen to have group structure. In the following, we investigate the permutations of an arbitrary constellation $\mathbf{c}_1$ that result in the same FER performance as $\mathbf{c}_1$: rotation, reflection, and multiplication by non-zero field elements.

A. Rotation

Now, consider the $q - 1$ constellations obtained by rotating the constellation points of $\mathbf{c}_1$ by $2k\pi/q$ for $k \in \mathbb{Z}_q\backslash\{0\}$. If instead of $\mathcal{M}$ the constellation mapper $\mathcal{M}' : \mathbb{F} \rightarrow \mathbb{C}$ that is equivalent to a $2k\pi/q$ rotation of the constellation induced by $\mathcal{M}$ is used, then the received constellation at the j-th time instant $\mathbf{M}_{\mathcal{M}'}(H_A(j), H_B(j))$ is identical to $\mathbf{M}_{\mathcal{M}}(H_A(j)e^{i2k\pi/q}, H_B(j)e^{i2k\pi/q})$. Since for Rayleigh fading the effect of the channel includes a random rotation uniform on $[0, 2\pi)$, the sets of possible received constellations $\mathbf{M}_{\mathcal{M}}(H_A(j), H_B(j))$ and $\mathbf{M}_{\mathcal{M}'}(H_A(j), H_B(j))$ have the same distribution. Therefore, any rotation $\pi_R^k\mathbf{c}_1$ of $\mathbf{c}_1$ has the same FER as $\mathbf{c}_1$ (here π_R is the rotation permutation).

B. Reflection

Also consider the constellation obtained by reflecting the constellation points of $\mathbf{c}_1$ on the x-axis. Let us denote by $\mathcal{M}' : \mathbb{F} \rightarrow \mathbb{C}$ the constellation mapper that is equivalent to the reflection of the constellation induced by $\mathcal{M}$ on the x-axis. Then at the j-th time instant, $\mathcal{M}'(S_A(j)) = \mathcal{M}(S_A(j))^*$ and $\mathcal{M}'(S_B(j)) = \mathcal{M}(S_B(j))^*$ and therefore

$$\begin{aligned}
&\mathbf{M}_{\mathcal{M}'}\left(H_A(j), H_B(j)\right)\\
&=\{H_A(j)\mathcal{M}'(S_A(j)) + H_B(j)\mathcal{M}'(S_B(j))|S_A(j), S_B(j) \in \mathbb{F}\}\\
&=\{H_A(j)\mathcal{M}(S_A(j))^* + H_B(j)\mathcal{M}(S_B(j))^*|S_A(j), S_B(j) \in \mathbb{F}\}\\
&=\{(H_A(j)^*\mathcal{M}(S_A(j)) + H_B(j)^*\mathcal{M}(S_B(j)))^*|S_A(j), S_B(j) \in \mathbb{F}\}\\
&= \mathbf{M}_{\mathcal{M}}\left(H_A(j)^*, H_B(j)^*\right)^*,
\end{aligned} \tag{9}$$

where for a set $A \subset \mathbb{C}$, $A^* = \{x^* | x \in A\}$. Since the added noise is complex-valued circularly symmetric AWGN and for Rayleigh fading the probability distributions of $H_i(j)$ and $H_i(j)^*$, for $i \in \{A, B\}$, are the same, a reflection (flip) $\pi_F\mathbf{c}_1$ of $\mathbf{c}_1$ has the same FER as $\mathbf{c}_1$ (here π_F is the reflection permutation).

C. Multiplication by Non-Zero Field Elements

Knowing that $\mathbf{S}_A$ and $\mathbf{S}_B$ are selected uniformly and iid on $\mathbb{F}^k$, encoding and transmitting $c\mathbf{S}_A$ and $c\mathbf{S}_B$ for $c \in \mathbb{F}\backslash\{0\}$ has the same FER performance as encoding and transmitting $\mathbf{S}_A$ and $\mathbf{S}_B$ if a linear code is used. This is because for $c = 1$, nothing has changed, while for $c \in \mathbb{F}\backslash\{0, 1\}$, we still have independent uniform distributions on $\mathbb{F}^k$. But compared to $c = 1$, the effect of using $c \in \mathbb{F}\backslash\{0, 1\}$ is as if we encoded using $c = 1$, and adopted the constellation $\mathbf{c}_2 = c\mathbf{c}_1 = (cc_{11}, cc_{12}, ..., cc_{1q}) = \mathbf{c}_1\pi_{\times c}$ instead of $\mathbf{c}_1$ at the relay. Therefore, any constellation resulting from multiplication of $\mathbf{c}_1$ by $c \in \mathbb{F}\backslash\{0\}$ has the same FER as $\mathbf{c}_1$ (here $\pi_{\times c}$ is the multiplication by c permutation).

D. Number of Distinct Mappers

Let $H_1 \subset \mathcal{P}$ be the set of permutations corresponding to rotations and reflections of the constellation (these are left permutations); and $H_2 \subset \mathcal{P}$ the set of permutations corresponding to multiplications by non-zero field elements (these are right permutations). For $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}$, if $h_1\mathbf{c}_1 h_2 = \mathbf{c}_2$, where $h_1 \in H_1$ and $h_2 \in H_2$, then $\mathbf{c}_1$ and $\mathbf{c}_2$ have the same performance in terms of FER and we say they are equivalent. The group $\mathcal{C}$ is thus divided into distinct classes of equivalent elements. If $\mathbf{c} \in \mathcal{C}$, we are interested in the double cosets $H_1\mathbf{c}H_2$ with respect to H_1 and H_2, as follows

$$H_1\mathbf{c}H_2 = \{h_1\mathbf{c}h_2 | h_1 \in H_1, h_2 \in H_2\}. \tag{10}$$

Thus, double cosets of H_1 and H_2 partition $\mathcal{C}$, where each partition consists of constellations with equivalent FER. The following theorem from [22] indicates the number of members in each partition, which in turn specifies the number of partitions and distinct mappers.

Theorem 2: For a double coset $H_1\mathbf{c}H_2$ and $\mathbf{c} \in \mathcal{C}$, the number of members in the partition with $\mathbf{c}$ is equal to

$$\#(H_1\mathbf{c}H_2) = \frac{|H_1||H_2|}{|H_1\mathbf{c} \cap \mathbf{c}H_2|}, \tag{11}$$

where $|.|$ indicates the size of the enclosed set. In the following, we investigate the number of distinct mappers for $GF(3)$, $GF(2^2)$, and $GF(5)$ fields, in detail.

For the case that $\mathbb{F} = GF(3)$, the group $\mathcal{C}$ has 6 elements and as $|H_1| = 6$, $\mathcal{C}$ has 1 partition. Thus even for non-linear coding schemes all 3! constellations have the same performance in terms of FER.

If $\mathbb{F} = GF(2^2)$, we have $|H_1| = 8$, $|H_2| = 3$, and $|H_1\mathbf{c} \cap \mathbf{c}H_2| = 1$ for all $\mathbf{c} \in \mathcal{C}$, and thus the group $\mathcal{C}$ will consist of one partition of size 24. Hence all $4!$ constellations have the same performance in terms of FER if a linear coding scheme is employed at the end nodes. If the code is non-linear, then the number of distinct mappers equals to $|H_2| = 3$.

Finally, if $\mathbb{F} = GF(5)$, we have $|H_1| = 10$, $|H_2| = 4$, and $|H_1\mathbf{c} \cap \mathbf{c}H_2| = 1$ or 2 depending on $\mathbf{c} \in \mathcal{C}$. Thus the group $\mathcal{C}$ will be partitioned into 4 groups: two of size 20, and two of size 40, which means that there are 4 distinct mappers if a linear code is employed at the end nodes. As a result, if linear coding is employed at the end nodes, only four distinct mappers should be considered as all others have identical performance in terms of FER. Fig. 1 depicts one member of each group. Note that Fig. 1(a) and (b) show constellations that belong to partitions of size 20, while Fig. 1(c) and (d) show constellations that belong to partitions of size 40.

V. A Lower Bound On the FER Performance

In this section, we find a lower bound on the FER performance for quasi-static Rayleigh fading channels. We denote by $\mathbf{S}_{AB} = a\mathbf{S}_A + b\mathbf{S}_B$ and by $\mathbf{U}_{AB} = a\mathbf{U}_A + b\mathbf{U}_B$. Let us denote by $\mathcal{I} = I(\mathbf{Y}_R; \mathbf{S}_{AB}|H_A, H_B)$, the mutual information between the superimposed received signals at the relay node and the decoded network-coding combination, given the channel gains. As the source messages are selected according to a uniform distribution,

$$\begin{aligned} nr\log_2(q) &= \log_2(q^k) \\ &= h(\mathbf{S}_{AB}) \\ &= h(\mathbf{S}_{AB}|H_A, H_B) \\ &= I(\mathbf{S}_{AB}; \mathbf{Y}_R|H_A, H_B) + h(\mathbf{S}_{AB}|\mathbf{Y}_R, H_A, H_B) \\ &\le \mathcal{I} + P_e nr\log_2(q) + 1 \qquad (12) \\ &= \mathcal{I} + n\epsilon_n, \qquad (13) \end{aligned}$$

where (12) follows from Fano's inequality, and $\epsilon_n \to 0$ as the probability of error $P_e \to 0$. According to (13) the probability of $\mathcal{I}$ being less than $nr\log_2(q)$ is an indicator of the probability of error, called *information outage probability*. We denote the information outage probability by $P_o = \Pr\{\mathcal{I} \le nr\log_2(q)\}$.

For a linear code, there is a one-to-one relation between $\mathbf{S}_{AB}$ and $\mathbf{U}_{AB}$, and thus $\mathcal{I} = I(\mathbf{Y}_R; \mathbf{U}_{AB}|H_A, H_B)$. For simplicity, in the following, we denote $Y_R(j)$, $S_{AB}(j)$, $U_{AB}(j)$, $U_A(j)$, and $U_B(j)$ for a time instant $j \in Z_n$ by Y_R, H_A, H_B, S_{AB}, U_{AB}, U_A, and U_B, respectively.

Now, assume a code of rate $r = k/n$ and, without loss of generality, that the first k symbols are systematic. Then the last $n-k$ symbols of $\mathbf{U}_A$ and $\mathbf{U}_B$ are dependent on the first k and hence the last $n-k$ symbols of $\mathbf{U}_{AB}$ are dependent to the first k. Let us denote the first k symbols of the received signal, by $\mathbf{Y}_R^{(1,k)}$ and the last $n-k$ by $\mathbf{Y}_R^{(k+1,n)}$. We denote the last $n-k$ symbols of the encoded packets by $U_A^{(k+1,n)}$ and $U_B^{(k+1,n)}$. The information $\mathcal{I}$ can then be rewritten as

$$\begin{aligned} \mathcal{I} &= h(\mathbf{Y}_R|H_A, H_B) - h(\mathbf{Y}_R|\mathbf{U}_{AB}, H_A, H_B) \\ &\le nh(Y_R|H_A, H_B) - h(\mathbf{Y}_R^{(1,k)}|\mathbf{U}_{AB}, H_A, H_B) \\ &\quad - h(\mathbf{Y}_R^{(k+1,n)}|\mathbf{Y}_R^{(1,k)}, \mathbf{U}_{AB}, H_A, H_B) \\ &\le nh(Y_R|H_A, H_B) - kh(Y_R|U_{AB}, H_A, H_B) \qquad (14) \\ &\quad - h(\mathbf{Y}_R^{(k+1,n)}|\mathbf{Y}_R^{(1,k)}, \mathbf{U}_{AB}, \mathbf{U}_A^{(k+1,n)}, H_A, H_B) \\ &= nh(Y_R|H_A, H_B) - kh(Y_R|U_{AB}, H_A, H_B) \\ &\quad - h(\mathbf{Y}_R^{(k+1,n)}|\mathbf{Y}_R^{(1,k)}, \mathbf{U}_{AB}, \mathbf{U}_B^{(k+1,n)}, \mathbf{U}_A^{(k+1,n)}, H_A, H_B) \\ &= nh(Y_R|H_A, H_B) - kh(Y_R|U_{AB}, H_A, H_B) - (n-k)h(Z_R) \\ &= kI(Y_R; U_{AB}|H_A) + (n-k)I(Y_R; U_A, U_B|H_B). \qquad (15) \end{aligned}$$

On the other hand, the information $\mathcal{I}$ can also be bounded by

$$\begin{aligned} \mathcal{I} &\le I(\mathbf{U}_{AB}; \mathbf{Y}_R, \mathbf{U}_A|H_A, H_B) \qquad (16) \\ &= I(\mathbf{U}_{AB}; \mathbf{Y}_R|\mathbf{U}_A, H_A, H_B) \\ &= h(\mathbf{Y}_R|\mathbf{U}_A, H_A, H_B) - h(\mathbf{Y}_R|\mathbf{U}_{AB}, \mathbf{U}_A, H_A, H_B) \\ &\le nh(Y_R|U_A, H_A, H_B) - nh(Z_R) \\ &= nI(Y_R; U_B|U_A, H_A, H_B) \qquad (17) \end{aligned}$$

Similarly,

$$\mathcal{I} \le nI(Y; U_A|U_B, H_A, H_B). \tag{18}$$

Thus from (15), (17), and (18), the information outage

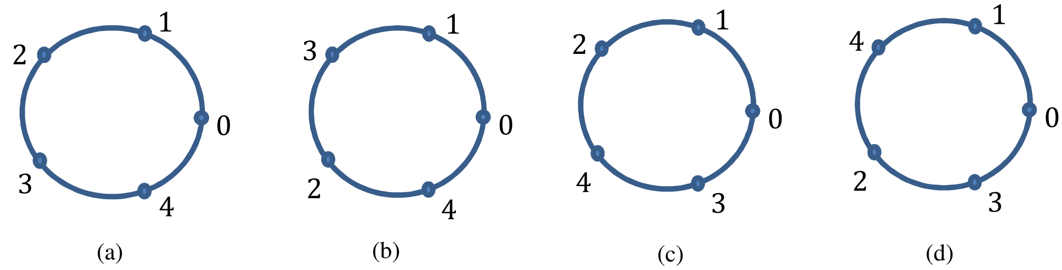

Fig. 1. The four distinct constellations: (a) belongs to a partition of size 20, (b) belongs to a partition of size 20, (c) belongs to a partition of size 40, and (d) belongs to a partition of size 40.

probability is lower bounded by

$$P_o > \Pr\{\min\{ rI(Y_R;U_{AB}|H_A,H_B)+(1-r)I(Y_R;U_A,U_B|H_A,H_B), I(Y_R;U_B|U_A,H_A,H_B), I(Y_R;U_A|U_B,H_A,H_B)\} \leq r\log_2(q)\}. \quad (19)$$

Knowing that the steps in (14) and (16) are potentially loose, we expect the final bound (19) to be unrealizable, i.e., we do not expect any actual code to meet it in terms of FER performance. The bound (19), is depicted in Fig. 4. Also, note that the bound in (19) is valid for each of the $GF(5)$ distinct constellations shown in Fig. 1.

VI. Simulation Results

In this section, we compare the performance of non-binary constellations in a PNC scenario with that of the binary case, using a practical concatenation of RS and convolutional codes, and LDPC codes.

For a fair comparison in terms of complexity, the number of states of the convolutional codes should be approximately the same. We choose the number of states to be 2^5, 3^3, 4^2, and 5^2 for $GF(2)$, $GF(3)$, $GF(2^2)$, and $GF(5)$ fields, respectively. The convolutional code rate is $1/2$. The best generator polynomial for each field was found by exhaustive search. Table I shows these generator polynomials.

Comparable Reed-Solomon codes should have the same input and output packet lengths, as well as the same rates. Here, we assume that the RS code rate is 0.8. Therefore, the concatenated RS-CC code rate is $0.8 \times 0.5 = 0.4$. Table I also provides the Reed-Solomon parameters used in this paper, where a RS (n,k,m) encodes k, m-symbol blocks into n blocks of size m symbols.

TABLE I
Concatenated RS-CC Parameters for Different Fields

Field	CC gen. poly.	RS (n, k, m)
$GF(2)$	(1 0 1, 0 1 1)	(63, 51, 6)
$GF(3)$	(2 0 1 1, 2 2 2 1)	(59, 47, 4)
$GF(2^2)$	(1 1 1, 1 α 1)	(63, 51, 3)
$GF(5)$	(1 1 3, 2 4 1)	(55, 45, 3)

A girth-12 LDPC code that has a parity check matrix of size 4395×7325 is used for the binary case. The number of

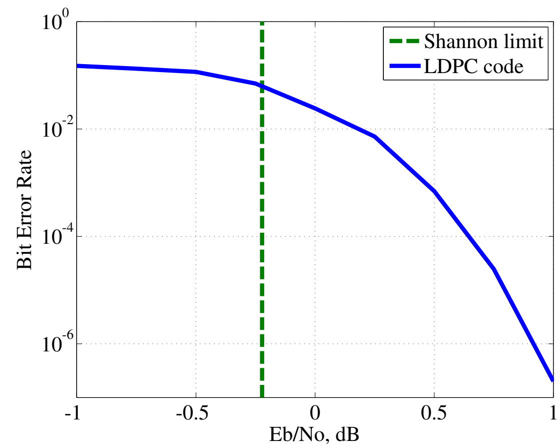

Fig. 2. Bit error rate of LDPC code and the Shannon limit at the rate 0.4

non-zero elements in each column is 3, and that in each row is equal to 5. The BER performance of this code is 0.8dB away from the Shannon bound at rate 0.4 (-0.22dB). Fig. 2 illustrates the BER performance of the LDPC code for the binary field over an AWGN channel. In Fig. 2, even at BER of 10^{-7} no error floor is observed, which is an indicator of a good parity check matrix design. For higher order fields, all non-zero entries in the binary matrix are randomly switched to non-zero elements of the higher order field, and in this way we construct all the LDPC parity check matrices in this paper.

Fig. 3 illustrates the performance of the binary and ternary RS-CC encoded in PNC scenarios for quasi-static Rayleigh fading channels. As can be seen from Fig. 3, attempting to decode against all choices of coefficients (a,b) decreases the frame error rate in a manner equivalent to a gain of approximately 1dB in E_b/N_0. Similarly for other fields, decoding against all choices of coefficients decreases the FER. This decrease in FER for higher order fields, i.e., $GF(2^2)$ and $GF(5)$, is greater since the number of choices of decoding coefficients (a,b) increases as the size of the field increases, according to *Theorem* 1.

In the following, all FER are obtained by attempting to decode according to all choices of valid coefficients (a,b). As can be seen from Fig. 4, thanks to the RS-CC error correcting code, higher order fields outperform the binary case by 1 to

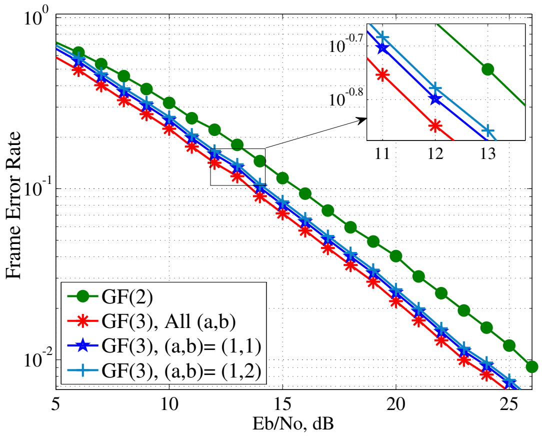

Fig. 3. Frame error rate for PNC configuration with RS-CC encoded packets over field $GF(3)$ when the relay attempts to decode against all decoding coefficients, i.e., (1,1) and (1,2) for quasi-static Rayleigh fading channels.

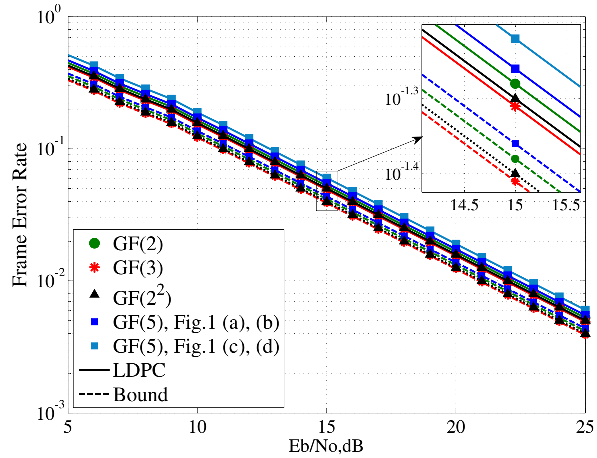

Fig. 5. The frame error rate for PNC configuration with LDPC coded packets for quasi-static Rayleigh fading channels

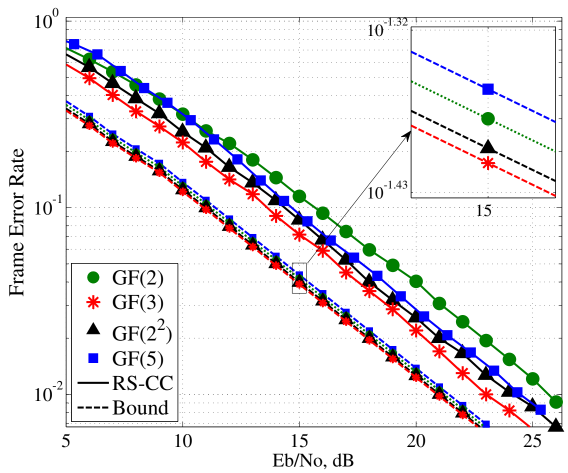

Fig. 4. The frame error rate for PNC configuration with RS-CC coded packets for quasi-static Rayleigh fading channels

3dB at FER of 10^{-2}, for quasi-static Rayleigh fading channels, with $GF(3)$ providing the best performance. The performance of RS-CC coded curves are approximately 2.5 to 4dB away from the lower bounds (19). For the selected RS-CC coding scheme, the four distinct constellations shown in Fig. 1 for $GF(5)$ have the same FER performance, hence their curves overlap in Fig. 4. Also from Fig. 4, the $GF(3)$ curve is the closest to its correspondence lower bound. As RS-CC coding results in a significant gap from the lower bounds, we also consider LDPC codes.

Fig. 5 illustrates the performance of binary and non-binary LDPC coding in PNC scenarios for quasi-static Rayleigh fading channels. Attempting to decode against all choices of coefficients (a, b) decreases the frame error rate by 0.2dB in E_b/N_0. Also, with LDPC coding at the end nodes, the FER performance is found to be only 1dB away from the bound (19). For the selected LDPC coding scheme, the constellation mappers shown in Fig. 1(a) and (b) have the same performance in terms of FER and they outperform the constellation mappers shown in Fig. 1(c) and (d) by about 0.1dB in E_b/N_0. The constellation mappers of Fig. 1(c) and (d) themselves appear to have identical FER performance. Fig. 5 also indicates that the non-binary fields, $GF(3)$ and $GF(2^2)$, outperform the binary case by 0.3dB and 0.1dB in E_b/N_0, respectively. However, the binary case has about 0.1dB performance gain compared to the field $GF(5)$. It should also be noted that fields $GF(3)$ and $GF(2^2)$ can only outperform the binary case by attempting to decode according to all coefficients (a, b), i.e., for a specific pair of decoding coefficients (a, b), binary coding leads to the best performance. Field $GF(5)$, however, performs worse than the binary case even after attempting to decode against all valid coefficients.

Fig. 6 illustrates the performance of the binary and non-binary RS-CC and LDPC codes in PNC scenarios for fast fading Rayleigh channels. For ease of presentation, we do not show the performance by decoding against one set of decoding coefficients in Fig. 6 and only the final FER achieved by attempting to decode against all valid coefficients are depicted. As can be seen form Fig. 6, using RS-CC coding, fields $GF(3)$ and $GF(2^2)$ outperform $GF(2)$ by 0.8dB and 0.5dB at FER of 10^{-4}, but only by taking advantage of decoding against all possible coefficients (a, b). Using LDPC coding, fields $GF(3)$, $GF(2^2)$ outperforms the binary case by 0.7dB at FER of 10^{-4}, but only by taking advantage of attempting to decode against all possible coefficients. However, in both coding schemes, the binary case outperforms the $GF(5)$ case. For fast fading Rayleigh channels, attempting to decode against all choices of coefficients (a, b) also significantly decreases the frame error rate providing a gain equivalent to 1 to 2dB in E_b/N_0 at FER 10^{-4}.

Finally, Figs. 5, 4, and 6 show that field $GF(3)$ has the best frame error rate performance among all considered fields, for both quasi-static Rayleigh fading and fast fading Rayleigh channels, employing either RS-CC or LDPC coding at the end nodes.

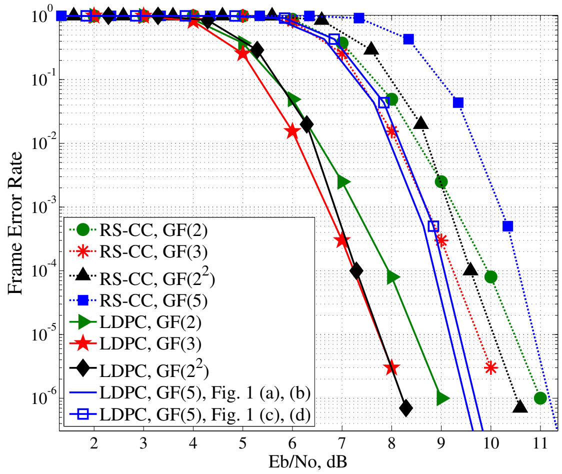

Fig. 6. The frame error rate for PNC configuration for fast fading Rayleigh channels

VII. Conclusion

In this paper, we have considered a problem of two-way wireless relaying, for which network coding is employed at the physical layer. The end nodes pick their symbols from a field $\mathbb{F}$ and transmit channel-coded PSK-modulated signals to the relay simultaneously. The relay node receives the superimposed channel-coded packets and directly decodes a network-coded combination of the source packets. The channel coding schemes employed are either a practical concatenation of RS-CC codes or LDPC codes. Working over non-binary fields allows the relay to attempt to decode different network-coded combinations, $aS_A + bS_B$ over $\mathbb{F}$, where $a, b \in \mathbb{F}\backslash\{0\}$. We have shown that for a finite field $\mathbb{F}$, there are effectively $|\mathbb{F}|-1$ unique network-coded combinations. We have investigated the performance of different constellation mappers for the finite field $\mathbb{F}$ in terms of FER. Exploiting the symmetric properties of the problem and using group theory arguments, we have shown that the number of constellation mappers with different performance is much less than the total constellation mappers $|\mathbb{F}|!$. Therefore, we do not need to consider all different constellation mappers. We have found a lower bound using Fano's inequality that confines the FER performance of decoding the network-coded combinations at the relay. Simulation results indicate that finite fields $GF(3)$, $GF(2^2)$, and $GF(5)$ outperform the binary case for quasi-static Rayleigh fading channels if RS-CC channel coding is performed at the end nodes. In addition, for Rayleigh fast fading channels, finite fields $GF(3)$ and $GF(2^2)$ outperform the field $GF(2)$ only by taking advantage of decoding according to all choices of a and b. Using LDPC codes for quasi-static Rayleigh fading and Rayleigh fast fading channels, finite fields $GF(3)$, and $GF(2^2)$ outperform the binary case only by taking advantage of attempting to decode multiple network-coded combinations. Finally, simulation results show that the finite field $GF(3)$ has the best probability of error performance among all considered fields.

As future work, it is of interest to consider non-binary constellations for larger systems in which more than two end nodes are communicating.

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