Adaptive Control of Linear Periodic Systems Using Multiple Models

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Abstract
The main thrust of the research at Yale in Adaptive Control, under the direction of the first author, and supported by NSF, has, in recent years, been on the hierarchical adaptive control of rapidly time-varying systems using multiple models. In this context, four distinct methods, developed in the past two decades by him and his graduate students using multiple fixed and adaptive models, are combined using a hierarchical approach.

During the course of the theoretical investigations, it was realized that the well known difficulties encountered in characterizing "rapidly time-varying parameters" also pose difficulties in deriving analytical proofs of stability in very general contexts. As a consequence, it was decided to study adaptive systems with periodic parameters, to gain greater understanding of the theoretical questions involved. This report, which is the first of a series, presents preliminary results on the problem of adaptive control of Linear Time-varying Periodic (LTP) Systems (i.e linear systems with periodically varying parameters).

1 Introduction

The adaptive control of linear time-invariant systems with unknown parameters have been studied extensively in the past five decades [1]. During the period 2005-2010, the first author with his graduate student Zhiling Tian, systematically extended the results to the adaptive control of linear systems with unknown periodic coefficients [2]-[3].
In all of them, the same approach that was used as in the LTI case [1], and adaptive identification and control problems were addressed. Linear periodic systems, which have been extensively investigated since the pioneering work of Floquet [4], when extended to adaptive cases where the periodic parameters are unknown, lend themselves to rigorous mathematical analysis. A variety of such problems which involve the identification of an unknown plant and controlling it to follow the output of a reference model have been treated by Narendra and Tian [2]-[3].

In the past two decades four distinct approaches (i) switching (ii) switching and tuning (iii) evolutionary adaptation and (iv) second level adaptation [5] have been proposed by the first author and his graduate students for adaptively controlling a plant using multiple models.

Research has been in progress at Yale for over a decade on combining these methods in adaptive control to obtain improved performance. Since 2015, supported by a National Science Foundation Grant, these efforts have been directed towards adaptive control of systems with "rapidly varying parameters". Since the latter are hard to characterize analytically, difficulties were encountered both at the problem formulation stage as well as deriving theoretical results for stability. It was therefore decided that specific classes of time-variations should be studied to gain a better understanding of the theoretical questions involved.

In this report, the adaptive control of linear periodic systems using multiple models is investigated. Theoretical results are derived for successively more complex cases and the responses of adaptive systems using a single model are compared with those using multiple models.

2 Static Systems

We start our investigations by considering the estimation of parameters in static systems, where the inputs and outputs are related by algebraic equations. In particular we shall consider the four subsystems:

(i) \( y(t) = \theta u(t), \quad \theta \in \mathbb{R}, u(t), y(t) \in \mathbb{R} \)

(ii) \( y(t) = \theta(t)u(t), \quad \theta(t) = \theta(t + T) \in \mathbb{R}, u(t), y(t) \in \mathbb{R} \)

(iii) \( y(t) = \theta^T u(t), \quad \theta \in \mathbb{R}^n, u(t) \in \mathbb{R}^n, y(t) \in \mathbb{R} \)

(iv) \( y(t) = \theta^T(t)u(t), \quad \theta(t) = \theta(t + T) \in \mathbb{R}^n, u(t) \in \mathbb{R}^n, y(t) \in \mathbb{R} \)
Cases (i) and (iii), where the unknown parameters are constant, have been well investigated in the adaptive control literature. We first consider (i) when $\theta$ is a constant briefly before attempting to extend results to the case where the unknown parameter $\theta(t)$ is a scalar periodic function of time. Similarly after discussing the case where $\theta \in \mathbb{R}^n$ is a constant vector, we consider the case where $\theta(t) \in \mathbb{R}^n$ is a periodic vector. As mentioned earlier, the adaptive control of linear periodic systems has been studied by Narendra and Tian using a single identification model in [2]-[3], and our main objective in this report is to investigate the difficulties encountered and the improvement in performance (if any), that can be achieved in such cases using multiple models. In Section 3, the static equations described in this section become parts of dynamical systems which are analyzed.

2.1 $y(t) = \theta u(t)$: $\theta$, $u(t)$, and $y(t) \in \mathbb{R}$

The input $u(t)$ and output $y(t)$ are related in this case by a linear equation where $\theta$ is an unknown constant parameter. To estimate $\theta$, a model is set up described by the equation

$$\dot{\hat{y}}(t) = \hat{\theta}(t)u(t)$$

(1)

Denoting $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$, $e(t) = \hat{y}(t) - y(t)$, we have the relation

$$e(t) = \tilde{\theta}(t)u(t)$$

The adaptive law

$$\dot{\hat{\theta}} = -e(t)u(t) = -u^2(t)\tilde{\theta}(t)$$

(2)

has been known since the 1960s. Using the Lyapunov function $V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^2(t)$, it follows that $\dot{V}(\tilde{\theta}) = \tilde{\theta}(t)\dot{\tilde{\theta}}(t) = -u^2(t)\tilde{\theta}^2(t)$. So $\tilde{\theta}(t)$ tends to zero if $u(t)$ is persistently exciting (in this case $u(t)$ is any function that satisfies the condition $\int_0^\infty u^2(\tau)d\tau = \infty$, or qualitatively, any input that does not tend to zero as $t \to \infty$).

If $\tilde{\theta} \to 0$, then $\hat{\theta} \to \theta$ where the convergence rate is controlled by the input $u(t)$.

**Comment 1:** While the analysis is carried out using equation (2), it is worth noting that the adaptive law is implemented using $\dot{\hat{\theta}}(t) = \hat{\theta}(t) = -e(t)u(t)$ (note that $\dot{\hat{\theta}} = \hat{\theta}(t)$ is valid only in those cases where the unknown parameters are constant).
Comment 2: Even in this very simple case where $\theta$ is a constant, it can assume any value, and this is one of the reasons why the time to estimate it may be long. If the bounds $\theta_{\min}$ and $\theta_{\max}$ are known the problem may be simplified to some extent depending upon how conservative these estimates are. The same questions assume significantly greater importance when $\theta(t)$ is time-varying as in case (ii).

Comment 3: If the adaptive law is modified to

$$\dot{\hat{\theta}}(t) = -\alpha e(t)u(t) \quad (3)$$

where $\alpha$ is a positive adaptive gain parameter, the speed of convergence of the estimate $\hat{\theta}(t)$ to the true value $\theta$ can be increased by increasing $\alpha$. In the absence of noise, arbitrarily fast convergence can be obtained by choosing $\alpha$ arbitrarily large (note that when no noise is present, $\theta$ can be estimated directly using the equation $\theta = \frac{y(t)}{u(t)}$).

Simulation 1: The convergence of $\hat{\theta}(t)$ when $u(t)$ is a random input with $\hat{\theta}(0) = 1$, and $\theta = 2$ is shown in Fig. 1 for different values of $\alpha$.

In Fig. 2 $\alpha = 1$ and the convergence of $\hat{\theta}(t)$ is depicted for different initial conditions $\hat{\theta}(0)$.

Comment 4: In Figures 2 and 3, adaptive models (1) with adaptive law (3) were used to estimate $\theta$. In these cases even fixed models are found to be adequate. If $\hat{\theta}$ assumes constant values in the entire domain in which $\theta$ can lie, the model closest to $\theta$ can be chosen using the amplitude of the error $e(t)$ at any instant. However, the constant $\hat{\theta}_{opt}$ chosen may not coincide with $\theta$ but may lie in its neighborhood. Adaptive models however result in zero parametric error asymptotically.
In the complex cases of interest to us in the following sections, a combination of fixed and adaptive models may have to be chosen for faster and more accurate convergence.

Comment 5: It is seen that the initial condition $\hat{\theta}(0)$ and the value of $\alpha$ both determine how rapidly $\theta$ can be estimated. In practice, practical considerations (such as the presence of noise) dictate the choice of $\alpha$. Hence, to improve the speed of response, multiple models with identical adaptive laws can be chosen with different initial conditions. Since the system is static and the performance criterion $e^2(t)$ is proportional to $\tilde{\theta}^2(t)$ at every instant, the model closest to the true value would be chosen as the best.

From the simple simulations presented, it is clear, and not surprising, that the initial estimate $\hat{\theta}(0)$, and the performance criterion $e^2(t)$ play an important role in
the choice of the optimum model.

Comment 6: If it is known a priori that $\theta_{\text{min}} \leq \theta \leq \theta_{\text{max}}$, and $N$ models are used to identify $\theta$ (with identical values of $\alpha$), the model closest to $\theta$ yields the fastest rate of convergence. This is shown in Fig. 2. It is seen that while $|\tilde{\theta}(t)|$ decreases monotonically as seen in Fig. 3a, $e^2(t)$ which depends upon the input is not monotonic in all cases (a typical response is shown in Fig. 3b).

2.2 $y(t) = \theta(t)u(t), \quad \theta(t + T) = \theta(t) \in \mathbb{R}$.

The difference in complexity, in estimating and controlling time-varying plants as compared to time-invariant plants is evident even at the level of a single time-varying parameter. In this section we attempt to extend some of the simple results presented in the previous section to the estimation of a scalar parameter $\theta(t)$ which is periodic with a known period $T$.

As in case (i) where the unknown parameter of $\theta$ was constant, multiple fixed values of $\theta$ can be used to estimate the time function $\theta(.)$ over an entire period. Assuming that $\theta_{\text{min}}$ and $\theta_{\text{max}}$, the bounds on $\theta(t)$, are known, $N$ fixed models can be used to determine the minimum value of $|e(t)|$ at every instant. Since $u(t)$ is common to all the models, the above procedure yields the model closest to $\theta(t)$ at that instant.

Simulation 2: If $\theta(t) = \sin(t)$ and $N$ is chosen to be 5, 20, and 100, respectively, the approximations of $\theta(t)$ given by the three sets of models are shown in Figs. 4a-4c. From Fig. 4c it is seen that the form of $\theta(t)$ can be determined quite accurately using a large number of fixed models.

Comment 7: Even though the approximate form of $\theta$ as a function of $t$ is obtained using multiple fixed models, the values are defined only at a finite set of points in parameter space. To obtain a continuous function $\hat{\theta}(t)$ that approximates sufficiently accurately, an adaptive model is needed. This model is given by difference equation

$$\hat{\theta}(t) = \hat{\theta}(t - T) - \frac{e(t - T)u(t - T)}{1 + u^2(t - T)} \quad (4)$$

where $\hat{\theta}(t)$ is updated from its value $\hat{\theta}(t - T)$, $T$ seconds earlier using the error and the input at that time. This equation is derived by minimizing the integral

$$V(\tilde{\theta}(t)) = \int_{t-T}^{t} \tilde{\theta}^2(\tau)d\tau$$

Simulation 3: Three periodic functions were approximated using the adaptive
law (4). The functions are respectively

\[ y(t) = \theta(t)u(t) \] where \( \theta(t) = 4 + \sin\left(\frac{2\pi t}{0.1}\right) + 3\cos^2\left(\frac{2\pi t}{0.1}\right) \]

\[ y(t) = \theta(t)u(t) \] where \( \theta(t) = 4 + \sin\left(\frac{2\pi t}{4}\right) + 3\cos^2\left(\frac{2\pi t}{4}\right) \)

\[ y(t) = \theta(t)u(t) \] where \( \theta(t) = 4 + \sin\left(\frac{2\pi t}{50}\right) + 3\cos^2\left(\frac{2\pi t}{50}\right) \)

which are low frequency, medium frequency, and high frequency signals. Furthermore, the input is considered to be a sinusoidal wave.

In each case the values are held constant over one period before initializing the adaptive process of the time-varying function. The behavior of \( \hat{\theta}(t) \) for small values of \( t \), as well as when the error \( \tilde{\theta}(t) \) over a period is small, are shown in Figs. 5a-5i for each of the functions approximated.

In all the simulations, roughly 12 measurements are needed at any instant over a period for sufficiently accurate estimates, and hence the convergence time is approximately the same.

**Simulation 4:** In the previous simulation it was assumed that there is no prior information concerning the unknown function \( \theta(t) \). If, as in simulation 2 fixed models are used as a first step in the estimation process and the information used in the adaptive model, the convergence is found to be significantly faster.
In this problem the objective is to estimate

\[ \theta(t) = 4 + \sin\left(\frac{2\pi t}{4}\right) + 3 \cos^2\left(\frac{2\pi t}{4}\right) \]

7 fixed models (i.e constant values of \( \theta \)) were distributed uniformly over the interval [2, 8] and their outputs were measured and compared with the outputs of the time-varying plant, at intervals of 0.1 units of time. The models that yielded an output error \( |e_i| \leq 1 \) over three periods were included in a set \( S \). The boundaries of the set as functions of time, are shown in Figure 6a. It is concluded that the desired function \( \theta(t) \) lies between these approximated curves (i.e the upper and lower bounds on \( \hat{\theta}(t) \) over a period are determined experimentally). The mean values of these bounds were chosen as initial conditions for the adaptive algorithms. The convergence of \( \hat{\theta}(t) \) to \( \theta(t) \) over the forth and tenth periods are shown in Figures 6b-6c respectively.
(a) Choice of initial conditions using multiple fixed models.
(b) Estimate $\hat{\theta}(t)$ and $\theta(t)$ in the interval $[3T, 4T]$.
(c) Estimate $\hat{\theta}(t)$ and $\theta(t)$ in the interval $[9T, 10T]$.

Figure 6: Performance of a single adaptive identifier for estimation of $\theta(t)$ using multiple fixed models for the choice of initial conditions; $\theta(t)$ blue curve and $\hat{\theta}(t)$ red curve.

2.3 $\theta \in \mathbb{R}^n$: $y(t) = \theta^T u(t)$, $\theta, u(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$.

We now proceed to estimate an unknown constant vector $\theta \in \mathbb{R}^n$. The adaptive law

$$\dot{\hat{\theta}}(t) = -e(t)u(t)$$

results in $\lim_{t \to \infty} \hat{\theta}(t) = \theta$ if $u(t)$ is persistently exciting. This is because the parametric error vector $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$ satisfies the equation

$$\dot{\tilde{\theta}}(t) = -u(t)u^T(t)\tilde{\theta}(t)$$
which has been studied extensively in the adaptive control literature (Morgan and Narendra 1977 [6]).

Simulation 5: The convergence of \( \hat{\theta}(t) \) to \( \theta \in \mathbb{R}^2 \) using the adaptive law (5) is shown in Figs. 7a-7c. In Figures 7a and 7b the convergence of \( \hat{\theta}_1(t) \) and \( \hat{\theta}_2(t) \) to the constant \( \theta_1 \) and \( \theta_2 \) are depicted as time functions. In Figure 7c, the convergence of \( \hat{\theta}(t) \) to \( \theta \) is shown in \( \mathbb{R}^2 \).

2.4 \( y(t) = \theta^T(t)u(t), \theta(t + T) = \theta(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}. \)

To estimate an unknown periodic parameter vector \( \theta(t) \), in a static system, the basic ideas developed in the previous cases are used. This implies the use of multiple models to determine the approximate shape of \( \theta(t) \) in \( \mathbb{R}^n \), and the choice of the initial condition \( \hat{\theta}(0) \) for use in the adaptive model.

The procedure used to estimate \( \hat{\theta}(t) \) in this case is similar to that used in case (ii)
for a single time-varying parameter $\theta(t)$. Using the Lyapunov functional candidate

$$V(\tilde{\theta}(t)) = \int_{t-T}^{t} \tilde{\theta}(\tau) \tilde{\theta}(\tau) d\tau$$

the law for determining the estimate $\hat{\theta}(t)$ can be derived as:

$$\hat{\theta}(t) = \hat{\theta}(t-T) - \frac{e(t-T)u(t-T)}{1 + u^T(t-T)u(t-T)}$$

whose form is the same as in the scalar case. Equation (6) can also be expressed as

$$\tilde{\theta}(t) = \left(E - \frac{u(t-T)u^T(t-T)}{1 + u^T(t-T)u(t-T)}\right) \tilde{\theta}(t-T)$$

(where $E$ is the unit matrix) and this implies that the error over the time sequence $t, t + T, \cdots$ decrease monotonically [6].

**Simulation 6:** The use of the adaptive law (6) with different initial conditions for a periodic $\theta(t)$ in $\mathbb{R}^2$ is shown in Fig. 8.

The speed of convergence is seen to depend critically on the initial condition chosen for $\hat{\theta}(0)$. Figs. 8a-8c and 8d-8f correspond to initial conditions $\hat{\theta}(0) = [0, 6]^T$ and $\hat{\theta}(0) = [7, 1]^T$, respectively. With $\hat{\theta}(0) = [0, 6]^T$ the convergence time is approximately 30 periods, while with $\hat{\theta}(0) = [7, 1]^T$ in the vicinity of the function

$$\theta(t) = [4 + \sin(0.5\pi t) + 3 \cos^2(0.5\pi t), \cos(0.5\pi t)]^T$$

at time $t = 0$ is seen to be approximately 20 periods. As seen in Figure 5 the differences between the two estimates increase with the frequency content of $\theta(t)$. Hence, as in the scalar case, it becomes essential to determine initial conditions that are close to the true trajectory $\theta(t)$, at time $t$.

As in case (ii), a small value of the error $\tilde{\theta}(t)$ results in a small error $e(t)$, but the latter can also be small due to the input $u(t)$ being orthogonal to $\theta(t)$. To avoid this (and taking into account the fact that $u(t)$ is persistently exciting) the errors are measured at intervals of $T$ to obtain consistent results. As in the scalar case, the approximate region in parameter space where $\theta(t)$ can lie is assumed to be known a priori (i.e. bounds on $\theta_i(t)$ are assumed to be known).

**Simulation 7:** The significant improvement in the convergence rate, when an initial estimate $\hat{\theta}(t)$ is available which is close to $\theta(t)$ (over a period $T$), is seen in the results of this simulation (shown in Fig. 9). Figures 9a-9d are the responses
Response for an arbitrary initial condition \( \hat{\theta}(0) = [7, 1] \) in (a,b,c).

Response for an arbitrary initial condition \( \hat{\theta}(0) = [0, 6] \) in (d,e,f).

Figure 8: Performance of the identifiers with different initial conditions.
obtained when no priori estimate of $\theta(t) = 0.1[5 + \sin(0.5\pi t), -1 - 0.5 \cos(0.5\pi t) - 0.5 \sin^2(0.5\pi t)]$ is available, and it is assumed that $\hat{\theta}_1(0)$ and $\hat{\theta}_2(0)$ are both zero over the first period $[0, T]$. Figures (b), (c), and (d) are respectively the estimates in the intervals $[10T, 11T]$, $[20T, 21T]$, and $[30T, 31T]$. While the function $\hat{\theta}(t)$ is close to $\theta(t)$ after 30 periods, the initial estimates up to 11 periods are seen to be poor. In contrast to this, when an initial estimate obtained with fixed models was used, convergence was rapid even in the first ten periods. This is shown in Figures (9e-9h).

In the following section dealing with dynamic systems, we shall indicate how these ideas can be used in identification and control problems.
9[a-d] with arbitrary initial condition. 9[e-h] with initial conditions chosen using fixed models.

Figure 9: Performance of the identifiers with and without initial estimates.
3 Dynamical Systems

Our principal interest in this report is in the identification and control of dynamical systems with periodic coefficients. While comments and conclusions made in Section 2 for static systems do not carry over directly to the dynamic case, they are nevertheless very useful in the choices made. In this section, we consider the following increasingly complex dynamical systems:

(a) $\dot{x} = a(t)x + bu$, $a(t + T) = a(t) \in \mathbb{R}$, $b \in \mathbb{R}$

(b) $\dot{x} = a(t)x + b(t)u$, $a(t + T) = a(t) \in \mathbb{R}$, $b(t + T) = b(t) \in \mathbb{R}$

and systems (a) and (b) are stable.

(c) $\dot{x} = A(t)x + bu$, $x(t) \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $(A(t), b)$ is in companion form and the system is stable.

(d) $\dot{x} = A(t)x + bu$, $x(t) \in \mathbb{R}^n$ and $A(t)$ and $b$ are as in (c), and $A(t + T) = A(t)$ is unstable.

In (a), (b), and (c) the objective is to identify the system. In (d), the objective is to identify and control the system so that the output $x(t)$ tracks the output of a stable linear time-invariant reference model.

Comment 8: While the systems chosen may appear to be very simple at first sight, they incorporate most of the difficulties that we will encounter in more complex problems in the future, involving time-varying parameters. As will become evident from the simulation studies included in this section, the ideas presented can be readily extended to such cases. Further, it is worth mentioning that problem (d) in which an unknown unstable time-varying system which has to be stabilized is a complex one, and the ideas presented in this context are applicable to more general cases.

3.1 $\dot{x} = a(t)x + bu$, $a(t + T) = a(t) \in \mathbb{R}$

In the scalar differential equation, $a(t)$ is periodic but the plant is known to be asymptotically stable. The objective is to set up a model and identify the parameter $a(t)$. The model is represented as

$$\dot{\hat{x}} = a_m \hat{x} + (\hat{a}(t) - a_m)x + bu; a_m < 0$$ (7)
which results in the error differential equation

\[ \dot{e} = a_m e + \tilde{a}(t)x(t) \quad (8) \]

where \( \tilde{a}(t) = \hat{a}(t) - a(t) \). The output error \( e(t) \) and the output of the plant \( x(t) \) are accessible, and the objective is to determine an adaptive law for adjusting \( \hat{a}(t) \) so that \( \lim_{t \to \infty} \tilde{a}(t) = 0 \). Using the Lyapunov functional

\[ V(e, \tilde{a}) = \frac{e^2}{2} + \frac{1}{2\alpha} \int_{t-T}^{t} \tilde{a}^2(\tau) d\tau \]

and choosing the adaptive law

\[ \hat{a}(t) = \hat{a}(t - T) - \alpha e(t)x(t) \quad (9) \]

it follows that

\[ \dot{V}(e, \tilde{a}) = a_m e^2 - \frac{\alpha}{2} e^2 x^2 \leq 0 \quad (10) \]

so that \( \lim_{t \to \infty} e(t) = 0 \). We note that in the dynamic context the parameter estimate \( \hat{a}(t) \) is adjusted after using the input and the error at that instant "\( t \)" (unlike the static case).

**Simulation 8:** The simple adaptive system with \( a_m = -5 \) was simulated for the case \( a(t) = -2 - 0.5 \cos(0.5\pi t) - 0.5 \sin^2(0.5\pi t), b = 5 \) which assures stability. The convergence of the adaptive parameter \( \hat{a}(t) \) to the true value of \( a(t) \) over the entire period is shown for three arbitrary initial conditions in Fig. 10. It is seen that the convergence rate critically depends upon the latter. Hence, appropriate initial conditions have to be determined in practice.

While the methods used in the static case are no longer directly valid in the dynamic case, approximations are possible using the integral of \( e^2(t) \) over one period,
Figure 11: Convergence of $\hat{a}(t)$ to $a(t)$ using multiple adaptive models and performance criterion $\int_{T}^{(i+1)T} e^2(\tau) d\tau$.

and choosing the values that correspond to the smallest values. This simulation is shown in Fig. 11.

3.2 $\dot{x} = a(t)x + b(t)u$, $a(t + T) = a(t)$, $b(t + T) = b(t) \in \mathbb{R}$

The problem is rendered significantly more complex when the two parameters $a(t)$ and $b(t)$ in the preceding equation are time-varying and periodic. Once again, as in the previous case, either fixed or adaptive models can be used, to speed up the convergence process. In the former, initial conditions $[\hat{a}(0), \hat{b}(0)]$ which result in fast convergence are determined. In the latter case, a set of adaptive laws with different initial conditions are initiated, and the best model is selected based on a performance criterion. The model is represented as

$$\dot{\hat{x}} = a_m \hat{x} + (\hat{a}(t) - a_m) x + \hat{b}(t)u; a_m < 0$$ (11)

which results in the error differential equation

$$\dot{e} = a_m e + \hat{a}(t)x(t) + \hat{b}(t)u$$ (12)

where $\bar{a}(t) = \hat{a}(t) - a(t)$ and $\bar{b}(t) = \hat{b}(t) - b(t)$. Using the Lyapunov functional $V(e, \hat{a}, \hat{b}) = \frac{e^2}{2} + \frac{1}{2a} \int_{T}^{T} \hat{a}^2(\tau) d\tau + \frac{1}{2b} \int_{T}^{T} \hat{b}^2(\tau) d\tau$ and choosing the adaptive laws

$$\hat{a}(t) = \hat{a}(t - T) - \alpha e(t)x(t)$$
$$\hat{b}(t) = \hat{b}(t - T) - \beta e(t)u(t)$$ (13)
it follows that \( \lim_{t \to \infty} e(t) = 0 \). These are illustrated in the following simulations.

**Simulation 9:** The adaptive system with \( a_m = -5 \) was simulated for the case \( a(t) = -2 - 0.5 \cos(0.5\pi t) - 0.5 \sin^2(0.5\pi t), b = -10 - 0.5 \sin(0.5\pi t)^2 \) which assures stability. The convergence of the adaptive algorithm to the true value of \( a(t) \) over the entire period is shown for arbitrary initial condition \( \hat{a}(0) = -2.5, b(0) = -11 \) in Fig. 12. As mentioned earlier, the convergence rate critically depends upon the initial conditions. Hence, appropriate initial conditions have to be determined in practice. Towards this end, multiple adaptive models were used and based on the behavior of \( \int_T^{(t+1)T} e^2(t) \, dt \) the best estimate was computed for each period.

### 3.3 Identification of a Stable LTP System

If a dynamical system is represented by the equation

\[
\dot{x} = A(t)x + bu(t)
\]

where \( x(t) \in \mathbb{R}^n \), \((A(t),b)\) are in companion form and \( A(t) = A(t + T) \in \mathbb{R}^{n \times n} \), it can be equivalently represented as

\[
\dot{x} = A_m x(t) + [A(t) - A_m] x + bu(t)
\]

where \( A_m \) is a time-invariant stable matrix. If a single model is used to identify it, we choose the form

\[
\dot{x} = A_m \hat{x}(t) + [\hat{A}(t) - A_m] x + bu(t)
\]
which results in the error equation

\[ \dot{e} = A_m e(t) + \Phi(t)x(t) \]

where \( \Phi(t) = \hat{A}(t) - A(t) \). By choosing the Lyapunov functional

\[
V(e, \Phi) = \frac{1}{2} e^T P e + \frac{1}{2} \text{Trace} \left( \int_{t-T}^{t} \Phi^T(\tau) P \Phi(\tau) d\tau \right)
\]

where \( P \) is the symmetric positive definite solution to the Lyapunov equation

\[ A_m^T P + P A_m = -Q; Q = Q^T > 0 \]

the adaptive law

\[ \Phi(t) = \Phi(t - T) - e(t)x^T(t) \]  \quad (14)

yield a negative semi-definite \( \dot{V}(e, \Phi) \) along any trajectory. This in turn assures that \( \lim_{t \to \infty} e(t) = 0 \). If the input \( u(t) \) is persistently exciting \( \Phi(t) \to 0 \) so that \( \hat{A}(t) \to A(t) \).

This problem is significantly more complex than the problems treated earlier due to the dimension of \( x(t) \). We assume in this case that the bounds on the parameter variations in \( A(t) \) are known so that identification can be carried out without the use of a very large number of models. In the following simulation, where \( A(t) \in \mathbb{R}^{2 \times 2} \), and only two parameters are unknown 8 models were used.

**Simulation 10:** Let

\[
A(t) = \begin{bmatrix}
0 & 1 \\
-4 + \sin(0.5\pi t) + 3 \cos^2(0.5\pi t) & \cos(0.5\pi t)
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
0 \\
1
\end{bmatrix},
\]

\[
A_m = \begin{bmatrix}
0 & 1 \\
-6 & -5
\end{bmatrix}
\]

and \( u(t) \) be white noise.

In the first experiment, four initial conditions were chosen randomly to emphasize that the convergence times can vary widely (see Fig. 13). In experiment 2, 25 models were initiated simultaneously and the model that minimized \( \int_{t(T)}^{(t+1)T} e^2(\tau) d\tau \) was considered as the best estimator (see Fig. 14).
Figure 13: Convergence of $\hat{a}(t)$ to $a(t)$ with different initial conditions.

Figure 14: Convergence of $\hat{a}(t)$ to $a(t)$ using multiple adaptive models.
3.4 Adaptive Control of an Unstable LTP System

In the previous Subsection we considered the identification of a stable LTP system. Since, all signals are bounded, our objective is primarily to assure parametric convergence in the shortest time. When the plant is unstable and the outputs of the plant can grow with time, the convergence time becomes even more critical. Methods which stabilize the system after the norm of the plant state vector has exceeded a specific value may not be practically feasible. Hence, the methods using multiple models proposed in this report become very relevant. The simulation considered in this section addresses such a problem of stabilization.

Simulation 11: A second order LTP system is described by the differential equation

\[ \dot{x}(t) = A(t)x(t) + bu(t) \]

where

\[ A(t) = \begin{bmatrix} 0 & 1 \\ 2 + \sin(0.5\pi t) + 3\cos^2(0.5\pi t) & \cos(0.5\pi t) \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

It can be verified (experimentally or theoretically) that the plant is unstable.

Comment 9: It is worth emphasizing that concluding the stability or instability of a periodically varying systems is considerably more difficult than that of a time-invariant system.

A reference model is defined by the differential equation

\[ \dot{x}_m(t) = A_m x_m(t) + br(t) \]

where

\[ A_m = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \]

is a constant stable matrix and \((A_m, b)\) is in companion form. The objective is to identify the plant and control it so that \(\lim_{t \to \infty} \|x(t) - x_m(t)\| = 0\).

If \(\hat{A}(t)\) is the estimate of \(A(t)\) at instant "\(t\)" and the last rows of the matrices \(A(t), \hat{A}(t),\) and \(A_m\) are respectively the time-varying vectors \(a^T(t)\) and \(\hat{a}^T(t)\), and the constant vector \(a_m^T\), the procedure described earlier can be used to identify the plant at every instant (i.e \(\hat{a}(t)\) as an estimate of \(a(t)\)) and choosing the input \(u(t)\) as:
\[ u(t) = r(t) + [\alpha_m^T - \hat{\alpha}^T(t)] x(t) \]

where \( r(t) = \sin(1.5t) + 2 \sin(3.5t) \). Figs. 15a and 15b illustrate performance of the controller using a single identifier with initial condition \([-9, 0]\), and Figs. 15c and 15d depict performance of the controller using multiple identifiers.

4 Conclusion

The problem of adaptively controlling rapidly time-varying systems is a very complex one. Defining the nature of the time-variations, and formulating appropriate
theoretical questions to be resolved, are both difficult. To make the problems theoretically tractable, the rapid adaptive control of linear systems with unknown periodically varying parameters was undertaken in this report. Even though only simple static and dynamic systems with one or two parameters were discussed, they nevertheless reveal the many decisions that have to be made to obtain viable solutions. In particular, it is shown that information from fixed and adaptive models have to be judiciously combined to choose initial conditions and adaptive laws. It is also worth pointing out that second level adaptation, which has proved very successful in time-invariant systems was not considered in this report since it cannot be directly extended to time-varying systems. Work is currently in progress, and the results will be discussed in future reports.

The class of systems treated in this report is limited to those with unknown parameters that are periodic with a known period $T$. Information of this nature is not available in general rapidly varying systems. However, it is clear from the results presented here that for adaptive control in general time-varying systems to be successful with multiple models, information from fixed and adaptive models must be generated and combined appropriately based upon the prior information available about the system.

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References


