Second Level Adaptation in Periodically Varying Environments

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Abstract

This is the second of a series of reports under preparation on the adaptive control of systems with rapidly time-varying parameters using multiple models and a hierarchical controller structure. The first two reports of the series are devoted to systems with periodically varying parameters. The objective is to study the complex questions that arise in the general case in a more structured setting. In the first report, the emphasis was on switching and tuning, using fixed and adaptive models. In this report, the same problems are investigated using second level adaptation. The results presented in the two reports indicate that the information obtained from different methods have to be suitably combined, based on the available prior information, to achieve the best performance of the overall system.

1 Introduction

Four methods for the adaptive control of dynamical systems using multiple models have been proposed by the first author and his students in the past 26 years. These include (i) Switching [1992] (ii) Switching and Tuning [1994-1997] (iii) Interactive/Evolutionary Adaptation [2003] and (iv) Second Level Adaptation [2012].

The first was introduced by Narendra and Balakrishnan at the 7th Yale workshop on Adaptive and Learning Systems in 1992. In the following years, this was refined by them to include adaptation, and the method is currently referred to as "Switching and Tuning" [5]. In 2003 Narendra and Feiler [6] proposed a new approach based
on multiple models known as "Interactive/Evolutionary Adaptation". Finally in
2012, Narendra and Han, in a series of papers [7, 8], proposed a new method called
second level adaptation. For approximately eight years, efforts have been in progress
at Yale to combine the different approaches using a controller with a hierarchical
structure to solve a much wider class of problems, and in particular those with
rapidly time-varying parameters.

The first two reports of this series deal with the adaptive control of systems with
unknown periodically varying parameters. Since the characterization of general
time-varying systems is complex, the class of systems considered was limited to those
with periodically varying parameters, to obtain better insight into the nature of the
difficulties that need to be addressed. In the first report the authors attempted
to combine switching, and switching and tuning, for adaptation. In the former,
multiple fixed models were used to determine suitable initial conditions from which
adaptive laws could be initialized. Numerous simulations were included to indicate
that the information from different sources have to be judiciously combined to
obtain the best results.

In this report, the second of the series, the adaptive control of periodically varying
systems is carried out using second level adaptation. At the end of the first report
it was mentioned that second level adaptation cannot be directly extended to such
systems. As seen in the following sections, several methods have to be combined to
make the approach successful.

Since an understanding of second level adaptation, as applied to the linear time-
invariant case, is essential to appreciate the difficulties encountered in the periodic
case, we summarize the principal assumptions and results in Section 2.

2 Second Level Adaptation: Linear Time-invariant
Plant

While Switching, as well as Switching and Tuning, have been very successful in prac-
tical applications, they nevertheless suffer from several shortcomings. The number
of fixed models needed is generally large, and at least one model has to be close
to the plant in parameter space. Further, very little information provided by all
the models is used explicitly in the decision process. Second level adaptation, in-
troduced in 2012 [7, 8], attempts to overcome these shortcomings. It is a radical
departure from earlier methods and uses significantly fewer models, all of whose
signals are used in the decision process. In the following subsections, the problem of identification of an unknown linear time-invariant stable plant is considered. In the following section the same problem is considered, where the plant has unknown periodically varying parameter.

2.1 The Method

An LTI plant $\Sigma_p$ is described by the state equation

$$\Sigma_p : \dot{x}_p = A_p x_p + bu$$

(1)

where $x_p \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A_p \in \mathbb{R}^{n \times n}$, and $(A_p, b)$ is in companion form. The last row of $A_p$ is $\theta_p^T$ and $\theta_p \in S_\theta$, where $S_\theta$ is a compact set in parameter space.

The Identification Problem: Assuming that the plant is stable and $\theta_p$ is unknown, the objective is to set up one or more models to estimate $\theta_p$ in a stable fashion.

The Control Problem: A reference model $\Sigma_m$ is described by the differential equation

$$\Sigma_m : \dot{x}_m = A_m x_m + br$$

(2)

where $A_m \in \mathbb{R}^{n \times n}$ is stable and $(A_m, b)$ is also in companion form and $r \in \mathbb{R}$ is a known bounded piecewise continuous function. Assuming that $A_p$ is unstable, the objective is to estimate $\theta_p$ as $\hat{\theta}_p(t)$ and use the estimate to control the plant so that all signals are bounded and $\lim_{t \to \infty} \|x_p(t) - x_m(t)\| = 0$ (this problem will be discussed in Section 3.5 for plants with periodically varying parameters).

2.1.1 The Identification Models

Expressing $\Sigma_p$ as

$$\Sigma_p : \dot{x}_p = A_m x_p + (A_p - A_m) x_p + bu$$

(3)

$n + 1$ stable identification models with series-parallel structure are chosen, where the $i^{th}$ model is described by

$$\Sigma_i : \dot{x}_i = A_m x_i + (A_i - A_m) x_p + bu$$

(4)

where $A_i$ is in companion form whose last row is a vector $\theta_i^T$ which is adaptively updated to provide an estimate of the plant parameter vector. Defining $e(t) =$
\(x_i(t) - x_p(t)\) and \(\tilde{\theta}_i(t) = \theta_i(t) - \theta_p\) the equation error can be expressed as

\[
\dot{e}_i = A_m e_i + b\tilde{\theta}_i^T x_p
\]  \hspace{1cm} (5)

### 2.1.2 Second Level Adaptation

It is assumed that \(S_p\) (and hence \(\theta_p\)) lies in the convex hull of the \(n + 1\) fixed vectors \(\theta_i\) \((i \in \Omega = \{1, 2, \ldots, n + 1\})\) defined earlier. This implies that \(n + 1\) constants \(\alpha_i\) exist such that

\[
\theta_p = \sum_{i=1}^{n+1} \alpha_i \theta_i
\]  \hspace{1cm} (6)

where \(\sum_{i=1}^{n+1} \alpha_i = 1\). Equation (6) represents an alternative parameterization of the plant parameter vector \(\theta_p\) and hence estimation of \(\alpha_i\) also implies the identification of the plant.

### 2.2 Adaptation

In the following we consider three distinct approaches to estimate \(\theta_p\). These are

i. estimating \(\theta_p\) adaptively using \(n + 1\) adaptive models starting from initial conditions \(\theta_i(0)\);

ii. estimating \(\alpha_i\) with \(\theta_i\) constant;

iii. simultaneously adapting both \(\theta_i(t)\) and \(\hat{\alpha}_i(t)\).

The conditions that need to be satisfied in each case are described below:

#### 2.2.1 Adaptive Adjustment of \(\theta_i(t)\)

If a single identification model is used to estimate \(\theta_p\), \(\theta_i(t)\) in equation (4)

\[
\Sigma_i : \dot{x}_i = A_m x_i + b\theta_i^T x_p + bu
\]  \hspace{1cm} (7)

is adjusted adaptively using the law

\[
\dot{\theta}_i(t) = -e_i^T P x_p
\]  \hspace{1cm} (8)
where $P$ is a positive definite matrix, which is the solution of the Lyapunov equation

$$A^T P + PA = -Q$$

The adaptive law (8) is derived using the Lyapunov function candidate $V(e_i, \tilde{\theta}_i) = e_i^T P e_i + \tilde{\theta}_i^T \tilde{\theta}_i$ where $\tilde{\theta}_i = \theta_i(t) - \theta_p$. The adaptive law (8) ensures that

$$\dot{V}(e_i, \tilde{\theta}_i) = -e_i^T Q e_i$$

(9)

where $Q = Q^T > 0$. This assures the boundedness of both the identification error $e_i$ and the parameter error $\tilde{\theta}_i(t)$, and the convergence of the parameter estimation error $\tilde{\theta}_i$ to zero if the input $u(t)$ is persistently exciting.

In second level adaptation all $n+1$ models are initialized at the constant vectors $\theta_1, \theta_2, \ldots, \theta_{n+1}$ and updated using identical adaptive laws (8).

In [7] it has been shown that at every instant "t", the unknown parameter vector $\theta_p$ lies in the convex hull of $\theta_i(t)$ and that the values corresponding to $\alpha_i$ remain constant. This is shown in Figs. 1-2.
2.2.2 Adaptive Adjustment of $\hat{\alpha}_i(t)$ ($\hat{\theta}_p(t) = \sum_{i=1}^{n+1} \hat{\alpha}_i(t) \theta_i$)

If the state error between the $i^{th}$ model described by equation (4) and the plant is $e_i(t)$, it follows from the fact $\theta_p = \sum_{i=1}^{n+1} \alpha_i \theta_i$ that

$$\sum_{i=1}^{n+1} \alpha_i e_i(t) = 0$$ (10)

Defining $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]^T$ and $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$. We obtain the equation:

$$\sum_{i=1}^{n} \alpha_i (e_i(t) - e_{n+1}(t)) = -e_{n+1}(t)$$ (11)

or

$$E_0(t)\bar{\alpha} = -e_{n+1}(t)$$ (12)

where columns of $E_0(t)$ are the vectors $[e_i(t) - e_{n+1}(t)]$. Theoretically, after the output errors of the $n + 1$ models are measured at a single instant, the value of $\bar{\alpha}$ can be computed. In practice, to make the procedure robust even in the presence of output noise, the following alternative procedure is adopted. The algebraic equation is replaced by a differential equation, whose solution tends asymptotically to the desired value. Premultiplying (12) by $E_0^T(t)$, we obtain the equivalent algebraic equation

$$E_0^T(t)E_0(t)\bar{\alpha} = -E_0^T(t)e_{n+1}(t)$$ (13)
whose solution $\bar{\alpha}$ is the desired vector. Expressing $\hat{\alpha}(t)$ as perturbation of $\bar{\alpha}$ i.e., $\hat{\alpha}(t) = \bar{\alpha} + \tilde{\alpha}(t)$, we get

\[
\dot{\hat{\alpha}}(t) = \hat{\alpha} = -E_0^T(t)E_0(t)\hat{\alpha}(t) - E_0^T(t)e_{n+1}(t) = -E_0^T(t)E_0(t)\tilde{\alpha}(t)
\]

which is uniformly asymptotically stable i.e., $\tilde{\alpha}(t) \to 0$ and $\hat{\alpha}(t) \to \bar{\alpha}$.

Figure 3a shows the evolution of the estimate of $\hat{\theta}_p(t)$. At $t = 0$, $\alpha_i = \frac{1}{3}$ and $\hat{\theta}_p = \sum_1^3 \theta_i(0)/3$. The simulations for two other values of $\theta_p$ (when they are close to one of the fixed parameters $\theta_i$) are shown in Figs. 3b and 3c. In Fig 4, the evolution of $\hat{\alpha}_i(t)$ are indicated for the three cases. Moreover, $\|\tilde{\theta}_p\|$ is illustrated in Fig. 5.
Figure 4: Evolution of $\hat{\alpha}_i(t)$ when only $\hat{\alpha}_i(t)$s are adjusted.

Figure 5: Evolution of $\|\tilde{\theta}_p(t)\|$ when only $\hat{\alpha}_i(t)$s are adjusted; blue curve $\theta_p = [-2, -2]^T$, red curve $\theta_p = [-4, -4]^T$, green curve $\theta_p = [0, -2]^T$. 

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2.2.3 Simultaneous Adaptation of $\theta_i(t)$ and $\hat{\alpha}_i(t)$

In Section 2.2.1 only $\theta_i(t)$ was adjusted, and at every instant the corresponding values of $\alpha_i(t)$ were computed so that $\sum_{i=1}^{3} \alpha_i(t)\theta_i(t) = \theta_p$. It was found that $\alpha_i(t)$ is the same constant for all values of $t$. In view of this, even though $\theta_i(t)$ ($i \in \Omega$) are being adapted, the desired value of $\hat{\alpha}_i$ can be computed algebraically.

Since our objective is to speed up the adaptive estimation of $\theta_p$, a natural extension is to adjust $\alpha_i(t)$ adaptively (as in Section 2.2.2) using newly acquired data concerning the errors $e_i(t)$. In the simple examples considered, the speed of convergence was invariably faster, even though the improvement in speed was not significantly greater than the approach considered in Section 2.2.2, where $\theta_i$ were held constant and $\alpha_i$ was adjusted adaptively.

It is also worth adding that the combined method described in Section 2.2.3 was observed (empirically) to be considerably faster than that considered in Section 2.2.1, when the region of uncertainty is small. The simulation results obtained for this case are presented in Figs. 6-8.

Comment: The above comments apply to cases when the unknown parameter $\theta_p$ is a constant. In the following section the three methods (explained in Sections 2.2.1-2.2.3) are extended to the case when $\theta_p(t)$ is a periodic function.
Figure 7: Evolution of $\hat{\alpha}_i(t)$ when both $\theta_i(t)$ and $\hat{\alpha}_i(t)$ are adjusted.

Figure 8: Performance of $\|\tilde{\theta}_p(t)\|$ when both $\theta_i(t)$ and $\hat{\alpha}_i(t)$ are adjusted.
3 Second Level Adaptation: Linear Periodic Time-Varying Plant

The importance of the convexity conditions is evident for the different cases described in Section 2, when the unknown parameter vector $\theta_p$ of the plant is a constant. In this section we attempt to extend the same methods to the case when $\theta_p(t)$ is periodic with period $T$, and $\theta_p(t)$ lies in the convex hull $\Delta_{n+1}$ of $n + 1$ constant vectors $\theta_i (i \in \Omega)$. Since $\theta_p(t)$ is periodic, it represents a closed curve in parameter space, and the above assumption implies that the entire curve lies in $\Delta_{n+1}$ for all "$t$". The objective is primarily to estimate the time-varying parameter exactly as $t \to \infty$, and to increase the speed of convergence wherever possible. As in the previous section we shall consider the following three cases:

i. $\theta_i (i \in \Omega)$ are adapted to estimate $\theta_p(t)$ (starting from $n + 1$ different initial conditions),

ii. $\theta_i (i \in \Omega)$ are fixed, but $\alpha_i(t)$ are adapted so that $\sum_{i=1}^{n+1} \alpha_i(t) \theta_i = \theta_p(t)$ and

iii. $\theta_i(t)$ and $\alpha_i(t)$ ($i \in \Omega$) are both adjusted adaptively to estimate $\theta_p(t)$.

3.1 Adaptive Adjustment of $\theta_i(t)$

As stated in report [10], the adaptive law

$$\theta_i(t) = \theta_i(t - T) - \gamma_1 e_i(t)^T P x_p(t)$$

(with $\gamma_1$ a positive constant) results in the convergence of the curve $\theta_i(t)$ to the given curve $\theta_p(t)$.

Comment: It is seen that the parameter vector $\theta_i(t)$ is updated from its value $T$ seconds earlier (i.e., $\theta_i(t - T)$), but using the error and the value of $x_p(t)$ at that instant of time, i.e., "$t$" [10].

To indicate the nature of convergence of $\theta_i(t)$ to $\theta_p(t)$, six simulations are included for six different initial conditions of the parameter vector $\theta_i(0)$. Due to space limitations, only the convergence of the first parameter $\theta_{11}(t)$ is indicated as a function of time in Fig. 9. It is clear that the speed of convergence is critically dependent on the initial conditions. Fig. 9f indicates that $\theta(0) = [-5, 0]^T$ results in faster convergence than initial conditions $\theta(0) = [20, 20]^T$, $\theta(0) = [15, 15]^T$ or $\theta(0) = [10, 10]^T$. 

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In two dimensions, the nature of convergence is shown for two initial conditions only. The forms of the curves are indicated over 100, 500, 1000, and 2000 periods (see Fig. 10). Once again it is obvious that the choice of initial conditions has a large effect on the speed of convergence.

Figure 9: Convergence of $\theta_1(t)$ to $\theta_{p1}(t)$ for $t \in [2000, 2020]$. 

(a) $\theta(0) = [20, 20]^T$. 

(b) $\theta(0) = [15, 15]^T$. 

(c) $\theta(0) = [10, 10]^T$. 

(d) $\theta(0) = [-10, 10]^T$. 

(e) $\theta(0) = [0, 0]^T$. 

(f) $\theta(0) = [-5, 0]^T$. 

Figure 9: Convergence of $\theta_1(t)$ to $\theta_{p1}(t)$ for $t \in [2000, 2020]$. 

(a) Initial condition $\theta(0) = [-5, 0]$. (b) Initial condition $\theta(0) = [10, 10]$. Figure 10: Convergence of $\theta(t)$ to $\theta^*_p(t)$ in $\mathbb{R}^2$ for two different initial conditions.
3.2 Adaptive Adjustment of $\hat{\alpha}_i(t)$

In Section 2.2, the problem of second level adaptation was briefly discussed when the unknown parameter of the plant $\theta_p$ is a constant vector and lies in the convex hull of $n+1$ constant vectors $\theta_i$. In such a case there exist constants $\alpha_i$ ($i \in \Omega$) such that

$$\theta_p = \sum_{i=1}^{n+1} \alpha_i \theta_i; \quad \sum_{i=1}^{n+1} \alpha_i = 1$$

or equivalently

$$\theta_p = \sum_{i=1}^{n} \alpha_i \left( \theta_i - \theta_{n+1} \right) + \theta_{n+1}$$

$$\theta_p = \Theta \bar{\alpha} + \theta_{n+1} \quad (17)$$

where $\Theta = [\theta_1 - \theta_{n+1}, \ldots, \theta_n - \theta_{n+1}]$ and $\bar{\alpha} = [\alpha_1, \ldots, \alpha_n]^T$.

In the present case $\theta_p(t)$ is a periodic function of time, with period $T$. Hence, functions $\alpha_i(t)$ exist such that

$$\theta_p(t) = \sum_{i=1}^{n+1} \alpha_i(t) \theta_i; \quad \sum_{i=1}^{n+1} \alpha_i(t) = 1$$

or equivalently

$$\theta_p(t) = \sum_{i=1}^{n} \alpha_i(t) \left( \theta_i - \theta_{n+1} \right) + \theta_{n+1}$$

$$\theta_p(t) = \Theta \hat{\alpha}(t) + \theta_{n+1} \quad (19)$$

The differential equations representing the plant and the model are respectively

$$\Sigma_p : \dot{x}_p = A_m x_p + b \theta_p^T(t) x_p + bu$$

$$\Sigma : \dot{x} = A_m x + b \left( \hat{\alpha}^T(t) \Theta^T + \theta_{n+1}^T \right) x_p + bu \quad (20)$$

where $\hat{\alpha}(t)$ is an estimation of $\bar{\alpha}(t)$.

The differential equation describing the error $e(t) = x(t) - x_p(t)$ is given by

$$\dot{e} = A_m e + b \left( \hat{\alpha}^T(t) - \bar{\alpha}^T(t) \right) \Theta^T x_p$$

(21)

Defining $\theta_i^T x_p(t) = z_i(t) \in \mathbb{R}$, the equation (21) can be expressed as

$$\dot{e} = A_m e + b \left( \hat{\alpha}^T(t) - \bar{\alpha}^T(t) \right) z(t)$$

(22)
where $z(t) = [z_1(t), \cdots, z_n(t)]^T$. The form of equation (22) is identical to that used in Section 2.1 and the adaptive law for adjusting $\alpha(t)$ can be given by inspection as

$$\hat{\alpha}(t) = \hat{\alpha}(t - T) - \gamma_2 e^T(t) Pbz(t)$$

(23)

where $\gamma_2 > 0$ is a constant adaptive gain.

The use of second level adaptation for identifying stable first and second order systems with periodic parameters are included in what follows.

**First Order Plant:** The unknown plant is described by the equation

$$\dot{x}_p = a_m x_p + b\theta_p^T(t)x_p + bu$$

(24)

where $a_m = -5$, $b = 1$, $\theta_p = 1 - \sin(2\pi t/4) - 3\cos^2(2\pi t/4)$, $b = 1$, and $u(t) = \sin(t) + \cos^2(1.5t)$.

In this simulation $\theta_1$ and $\theta_2$ are chosen as 10 and $-10$, respectively. The estimations of $\alpha_1(t)$ and $\alpha_2(t)$ are shown in Figs. 11. As in the time-invariant case discussed in Section 2, $\alpha_i(t)$ can be considered as an alternative representation of the systems. If $\hat{\alpha}_i(t)$ are the estimates of $\alpha_i(t)$, they can be used to determine $\hat{\theta}_p(t)$, the estimate of the periodic parameter $\theta_p(t)$ (see Fig. 12).

**Second Order Plant:** The plant, in this case, is described by the differential equation

$$\dot{x}_p = A_m x_p + b\theta_p^T(t)x_p + bu$$

(25)

where $x_p \in \mathbb{R}^2$, $\theta_p(t) = \theta_p(t - T) \in \mathbb{R}^2$,.
Using the procedure outlined in this subsection, $\hat{\alpha}_1(t)$, $\hat{\alpha}_2(t)$ are adjusted and their evolution is shown in Fig. 13. The corresponding values of $\hat{\theta}_p(t)$ and its convergence to $\theta_p(t)$ are shown in Figs. 14 and 15. Note that in this simulation, $\theta_i$s were selected as $\theta_1 = [10, -2]^T$, $\theta_2 = [-10, -2]^T$, and $\theta_3 = [-10, 2]^T$.

### 3.3 Approximate Estimation of $\theta_p(t)$

In the previous subsection, we considered the exact estimation of the unknown periodic parameter $\theta_p(t)$, by suitably parameterizing the model. In some cases, the speed of estimation may also be an important consideration. In this subsection we propose a procedure which converges more rapidly than in the previous case, but to an approximate solution. To obtain more accurate convergence, the system then switches to an adaptive model similar to that described in Section 3.1.

**The Procedure:** In Section 2, while dealing with a constant unknown parameter $\theta_p$, the equation

$$
\sum_{i=1}^{n+1} \alpha_i e_i(t) = 0
$$

(26)
Figure 13: Evolution of $\hat{\alpha}_i(t)$, $i = 1, 2, 3$.

Figure 14: $\int_{\theta}^{\theta + 1} ||\bar{\theta}_p(\tau)||d\tau$ using multiple model based approach.
was used to estimate the parameters $\alpha_i$. A similar procedure may be adopted when the parameter varies with time. Since $\theta_p(t)$ is embedded in a dynamical system, equation (26) is no longer strictly valid. However, it is found to be adequate to obtain an approximate estimate $\hat{\theta}_p(t)$ of $\theta_p(t)$.

As described earlier, in the time-invariant case $\alpha_i$ can be estimated either algebraically or by using a differential equation. By Using equation (26), the adaptive laws can be expressed as

$$\hat{\alpha}(t) = \hat{\alpha}(t - T) - E_1^T(t - T) \left( \hat{l}(t - T) - l(t - T) \right) \frac{1}{1 + \|E_1(t - T)\|^2}$$

(27)

where $E_1(t - T) = [e_1(t - T) - e_{n+1}(t - T), \ldots, e_n(t - T) - e_{n+1}(t - T)]$, $l(t - T) = -e_{n+1}(t - T)$, and $\hat{l}(t - T) = E_1(t - T)\hat{\alpha}(t - T)$.

The following procedure, based on the latter, is used in the simulation described below:

A plant is described by the second order differential equation

$$\dot{x}_p = A_m = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\theta^T_p(t)x_p + bu)$$

(28)

where $\theta_p(t) = \left[ -2 + \sin(0.5\pi t) + 3\cos^2(0.5\pi t) \cos(0.5\pi t) - 5 \right]^T$. It must be noted that it is not immediately evident whether this plant is stable or unstable. We assume in this study that prior information is available that the plant is indeed stable (so that no control is needed). The parameters of the multiple model based approach are initialized at $\theta_1(0) = [-10, -10]^T$, $\theta_2(0) = [10, 10]^T$, and $\theta_3(0) = [-10, 10]^T$ and the signle model is initialized at $\theta(0) = \sum_1^3 \theta_i(0)\alpha_i(0) = \sum_1^3 \theta_i(0)/3 = [-10/3, 10/3]^T$. The advantage of using the multiple model based
approach over that using a single model is evident in the simulations reported in Figs. 16 and 17.

3.4 Simultaneous Adaptation of $\theta_i(t)$ and $\hat{\alpha}_i(t)$

In Section 3.1, the parameters of models were adjusted adaptively so that all of them would converge asymptotically to the periodically varying parameter $\theta_p(t)$. In Section 3.2, time-varying parameters $\hat{\alpha}_i(t)$ were adjusted adaptively, assuming that $\theta_i$ are constant vectors.

In this case we attempt an approximate procedure where both $\hat{\theta}_i(t)$ and $\hat{\alpha}(t)$ are adjusted. Using the single model proposed in Section 3.2, i.e.,

$$
\Sigma : \dot{x} = A_m x + b \left( \hat{\alpha}^T(t) \Theta^T + \theta^T_{n+1} \right) x_p + bu
$$

(29)

The set of $2n + 2$ parameters $\alpha_i(t)$ and $\theta_i(t)$ ($i \in \Omega$) are adjusted as follows:

$$
\theta_i(t) = \theta_i(t - T) - \gamma_1 e^T P b x_p(t)
$$

(30)

$$
\hat{\alpha}(t) = \hat{\alpha}(t - T) - \gamma_2 e^T(t) P b z(t)
$$

(31)

The simulation of a second order system, using the above approach is shown in...
(a) Single model based approach.  
(b) Multiple model based approach.

Figure 17: Convergence of $\hat{\theta}_p(t)$ and $\theta_p(t)$ in $\mathbb{R}^2$. 
the following.

Consider the following second order dynamical system:

\[
\dot{x}_p = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\theta^T_p(t)x_p + u)
\]

where

\[
\theta_p(t) = \begin{bmatrix} 4 + \sin(2\pi t/4) + 3 \cos^2(2\pi t/4) \\ -0.5 + 0.5 \sin(2\pi t/4) + \cos(2\pi t/4) \end{bmatrix}
\]

The initial conditions of second level adaptation based approach are chosen as \(\theta_1(0) = [10, -2]^T\), \(\theta_2(0) = [-10, -2]^T\), and \(\theta_3(0) = [-10, 2]^T\), and the parameters of the single model based scheme is initialized at \(\theta(0) = \sum_1^3 \theta_i(0)\alpha_i(0) = \sum_1^3 \theta_i(0)/3 = [-10/3, -2/3]^T\). Simulation results are depicted in Figs. 18 and 19. From these figures, it is clear that multiple model based identification strategy converges after 20 periods. However, the approach using a single model results in a performance which is not comparable to it even after 400 periods.

### 3.5 Adaptive Stabilization

In the preceding four subsections we considered the identification (estimation) of the unknown periodically varying parameters of a linear plant using input and
Figure 19: Convergence of $\hat{\theta}_p(t)$ to $\theta_p(t)$ in $\mathbb{R}^2$. 

Single model based approach. 

Multiple model based approach.
state information. While accurate estimation of the parameters was the principal objective, the speed of convergence was also emphasized at every stage. For the identification of a stable plant, the latter, while attractive, is not always a critical consideration. However, when the plant is unstable, the rapidity with which the parameters can be estimated becomes extremely important in developing practically applicable methods. In view of this, the methods developed in Subsection 3.1-3.4 are applied in this section to stabilize unstable systems with unknown periodic parameters. Since the necessary theory has been developed, only the simulation results are presented in this section.

Consider the following dynamical system

\[
\dot{x}_p = A_p(t)x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\theta_p^T(t)x_p + u)
\]

\[
A_p(t) = \begin{bmatrix} 0 & 1 \\ 2 + \sin(0.5\pi t) + 3\cos^2(0.5\pi t) & 1 + 2\sin^2(0.5\pi t) \end{bmatrix}
\]

The above system can be shown to be unstable either theoretically (using Floquet theory) or experimentally. The latter procedure was used in this case.

The objective is to force the system states to track states of the following stable reference model:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r
\]

Define \( u = r + [-6, -5]x - \hat{\theta}_p(t)x \), where \( \hat{\theta}_p(t) \) can be obtained using a single or multiple model based approach. Note that in all of the simulations, \( \theta_i(t) \) is obtained using the normalized version of adaptive law (30) [7].

Simulation results are presented in Figs. 20-25. In Figs. 20, 22, and 24 the convergence of the parameters are shown, while in Figs. 21, 23, and 25 the corresponding state estimation errors are indicated. These three sets of figures correspond to three different initial conditions of the parameters which are given below:

1. \( \theta_1(0) = [6, 10]^T \), \( \theta_2(0) = [6, -10]^T \), and \( \theta_3(0) = [-20, -10]^T \) for second level adaptation and at \( \theta(0) = \sum_3 \theta_i(0)\alpha_i(0) = \sum_1 \theta_i(0)/3 = [-2.66, -3.33]^T \) for single model.

2. \( \theta_1(0) = [6, 10]^T \), \( \theta_2(0) = [6, -10]^T \), and \( \theta_3(0) = [-10, -10]^T \) for second level adaptation and at \( \theta(0) = \sum_3 \theta_i(0)\alpha_i(0) = \sum_1 \theta_i(0)/3 = [0.66, -3.33]^T \) for single model.
3. $\theta_1(0) = [6, 10]^T$, $\theta_2(0) = [6, -5]^T$, and $\theta_3(0) = [-5, -5]^T$ for second level adaptation and at $\theta(0) = \sum_{i=1}^{3} \theta_i(0) \alpha_i(0) = \sum_{i=1}^{3} \theta_i(0)/3 = [2.33, 0]^T$ for single model.

Since the plant is unstable, starting the system with a large initial parameter error results in extremely large output errors, using a single model, as seen in Fig. 21. Even in this case, the multiple model based approach is seen to result in a small error (maximum error of 0.2). However, this comparison is not realistic, since a designer, using a single model to stabilize an unstable system, would be considerably more cautious in the choice of the parameters.

In Figs. 22 and 23, the convergence of both parameters and output errors for a more realistic situation is depicted. Due to the slow convergence rate of the parameter using a single model, the maximum output error is seen to be three orders of magnitude larger than that observed using multiple models.

Finally, in Figs. 24 and 25, while both single and multiple model based performances may be practically acceptable, the latter is seen to have an output error which is a fraction of the former.

The above simulations conclusively demonstrate that the multiple model based approach is far superior to that based on a single model for the stabilization of an unknown plant that is unstable with periodic parameters.

![Figure 20](image-url)  
**Figure 20:** $\int_0^{(i+1)T} \|\hat{\theta}_p(\tau)\| d\tau$ using single and multiple model based controllers (red and blue curves respectively).
4 Conclusion

Second level adaptation, which was shown to be very successful for the identification and control of linear time-invariant plants with constant unknown parameters in [7], is extended in this report to the time-varying case where the unknown parameters are periodic. Both exact and approximate methods are proposed. Even in the latter case, the redeeming feature is that the identifier can switch to a single (accurate) adaptive model, when the estimation error is small. This is reminiscent of the switching and tuning concept which has been popular for over two decades.

As stated in the introduction, the first two reports of this series deal with unknown periodically varying parameters, to obtain better insight into the nature of difficulties encountered in more complex time-varying situations. It is evident that even in this limited class of systems, information from different sources have to be used to achieve satisfactory performance.

The simulation study presented in Section 3.5 of stable adaptation of an unstable plant shows clearly the superiority of multiple model based approaches in critical situations.

In the following reports more general time-varying situations will be considered to determine the efficacy of approaches based on multiple models.
Figure 22: $\int_{T}^{(i+1)T} \| \tilde{\theta}_p(\tau) \| d\tau$ using single and multiple model based controllers (red and blue curves respectively).

(a) The tracking error using single model based approach.  
(b) The tracking error using multiple model based approach.

Figure 23: $\| x - x_M \|$ using single and multiple model based controllers.
Figure 24: $\int_{0}^{T} (i+1) \| \theta_p(\tau) \| d\tau$ using single and multiple model based controllers (red and blue curves respectively).

(a) The tracking error using single model based approach. (b) The tracking error using multiple model based approach.

Figure 25: $\| x - x_M \|$ using single and multiple model based controllers.
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References


