Stabilizing a Multi-Agent System to an Equilateral Polygon Formation

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Abstract— **The problem of stabilizing a group of agents in the plane to a stationary formation is analyzed. A local control scheme is proposed to stabilize the agents to the vertices of an equilateral polygon. The centroid of the agents is stationary during the evolution. For three agents a full stability analysis is performed: If three agents start in distinct and non-collinear locations, they converge to the vertices of a stationary equilateral triangle, while maintaining a stationary centroid.**

I. Introduction

There has been a considerable amount of work on formation stabilization in the multi-agent systems literature. The two main types of formation stabilization that are studied are stabilization to a moving formation, and stabilization to a stationary formation. Much of the work has been on stabilizing a group of agents (most commonly unicycles) to a moving formation. For example, Justh and Krishnaprasad [1] develop a control law to stabilize two unicycles moving at constant speed to a common heading. In [8] by Paley et al., the same problem is approached by using potential functions to maintain spacing between unicycles. Marshall et al. [5] create a control strategy based on cyclic pursuit in which the unicycles can converge to a circle formation. In formation the unicycles are moving around the circle, equally spaced. In [11], [12] by Tanner et al., moving formations are studied in the context of flocking. The agents are modeled as double integrators (i.e., the control input is the acceleration) and the stability of a flocking control law is studied for both fixed and dynamic communication topologies.

In the area of stabilization to a stationary formation there are some interesting results. Sugihara and Suzuki [10] propose a heuristic distributed algorithm to stabilize a group of agents (modeled as point masses) to stationary positions, equally spaced around a circle. Each agent adjusts its position based on the position of the nearest agent and the farthest agent. Through simulation, it is shown that the agents form a rough approximation of a circle. The formation stabilization problem has also been studied using graph theory, as in, for example, [6], [7]. In this work a formation is viewed as a rigid graph, where the links on the graph represent the distance constraints between agents. Problems such as determining the best way to split a large rigid formation into smaller rigid formations are studied in this framework.

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Another scheme for formation stabilization of point masses is given by Lin et al. [3]. This scheme requires that each agent be equipped with a compass, so that they share a common direction. If the agents have this property then a local control strategy can be designed to stabilize to any stationary formation. In [4] this idea is extended to unicycles.

In this paper we look at the problem of stabilizing a group of agents to a stationary formation. We model the agents as point masses, and we uniquely identify each of the n agents with a number between 1 and n. The position of the i^{th} agent is given by the vector $z_i = (x_i, y_i)$ in $\mathbb{R}^2$. The input to each agent is a velocity vector u_i:

$$\dot{z}_i = u_i.$$

The agents' positions can also be represented as points in the complex plane $z_i = x_i + jy_i$, $i = 1, \ldots, n$. The agents are not equipped with a compass, and thus they do not share a common heading (i.e., the agents are disoriented). The fact that the agents are disoriented makes formation stabilization significantly more complicated then when the agents are oriented. The problem we address is to find a local control strategy such that for each i, agent i is stabilized to a distance $b > 0$ from agents $i+1$ and $i-1$.

The organization of the paper is as follows. In Section II we briefly examine the strategy in [3] in order to contrast with the present scheme. In Section III we introduce the control strategy, which is based on the linear polygon shortening scheme of [9], and analyze the system for n agents. Finally, in Section IV we study the special case of three agents and show that they stabilize to an equilateral triangle.

II. Formation stabilization with a compass

Consider a strategy in which each agent pursues a displacement of the next

$$\dot{z}_i = (z_{i+1} + d_i) - z_i, \quad i = 1, ..., n,$$

where the index i is evaluated modulo n and $\sum_{i=1}^{n} d_i = 0$. In vector form, this can be written as $\dot{z} = A_1 z + d$, where $A_1 = \mathrm{circ}(-1, 1, 0, \ldots, 0)$. A result from [3] is that the centroid of $z_1(t), \ldots, z_n(t)$ is stationary, and there exists a unique vector h orthogonal to $\ker A_1$ such that $A_1 h + d = 0$. Every $z_i(t)$ converges to the stationary centroid displaced by h_i.

By appropriate choice of d, a group of agents can be stabilized to a formation about their centroid. For example, let

$$d_i = \mathrm{e}^{2\pi i j/n}.$$

Notice that $\sum_{i=1}^n d_i = 0$ and therefore the centroid of the n points is stationary. In equilibrium $z_{i+1} - z_i = \mathrm{e}^{2\pi i j/n}$. Therefore, this stabilizes a group of agents to the vertices of a regular polygon centered at the centroid. However, notice that in order to implement this scheme, each agent must be able to calculate the vector $\mathrm{e}^{2\pi i j/n}$. This vector resides in the global coordinate system, which in this case is a global complex plane. Therefore, in order to implement this scheme, each agent must agree on a real and imaginary axis. Hence, each agent must be equipped with a compass. In this paper, the agents are not equipped with compasses. This makes the problem considerably more difficult.

III. Local control scheme and stability analysis

Consider a group of n agents, numbered from $1, \ldots, n$, lying in the plane. The i^{th} agents' position is given by (x_i, y_i), which we can represent in the complex plane as $z_i = x_i + jy_i$. We can view the group of agents as the vertices of an n-gon by joining consecutive pairs of points $z_1, z_2, \ldots, z_n$ to create the sides $z_1z_2, z_2z_3, \ldots, z_nz_1$. In this section we will introduce a control scheme for stabilizing the agents to an equilateral n-gon and study the stability of its equilibria.

A. The z dynamics

In order to stabilize a group of agents to an equilateral polygon, consider the following control strategy:

$$\begin{aligned}\dot{z}_i = u_i = \frac{1}{2}(z_{i+1} - z_i)\left(1 - \frac{b^2}{|z_{i+1} - z_i|^2}\right) \\ + \frac{1}{2}(z_{i-1} - z_i)\left(1 - \frac{b^2}{|z_{i-1} - z_i|^2}\right), \quad i = 1, \ldots, n,\end{aligned} \tag{1}$$

where b is a positive constant. In this expression all indices are evaluated modulo n (i.e., $n+1 = 1$ and $0 = n$). To better understand the motivation behind this scheme, consider the first term on the right-hand side of (1). If $|z_{i+1} - z_i| > b$ then $1 - b^2/|z_{i+1} - z_i|^2 > 0$ and thus the agent moves towards z_{i+1}. Similarly, if $|z_{i+1} - z_i| < b$ then $1 - b^2/|z_{i+1} - z_i|^2 < 0$ and the agent moves away from z_{i+1}. Therefore, the effect of this term is to stabilize z_i to a distance b from z_{i+1}. We add the second term to the right hand-side of (1) (which stabilizes z_i to a distance b from z_{i-1}) so that the centroid will remain stationary throughout the evolution.

Notice that if $b = 0$ we simply have the linear polygon shortening scheme of [9]. Also notice that the system is undefined if $|z_{i+1} - z_i| = 0$ for some i. Letting $z \in \mathbb{C}^n$ denote the $n \times 1$ vector of positions, $(z_1, \ldots, z_n)$, the system (1) is defined on the set

$$\mathcal{T} := \{z \in \mathbb{C}^n : |z_{i+1} - z_i| > 0, \ \forall i\}.$$

The system (1) has been chosen to stabilize to the configuration $|z_{i+1} - z_i| = b$, $\forall i$. However, it is difficult to study the stability of this formation in the z dynamics, since z_{i+1} and z_i could be going off to infinity together, and yet $|z_{i+1} - z_i|$ could be converging to b. Because of this, we introduce the notation $e_i = z_{i+1} - z_i$ and study the stability of the e dynamics with respect to the equilibrium $|e_i| = b$, $\forall i$. From this analysis we will be able to infer the stability of the z dynamics.

B. The e dynamics

We introduce the notation

$$e_i = z_{i+1} - z_i.$$

Notice that by the definition of e_i,

$$\sum_{i=1}^n e_i = 0. \tag{2}$$

Let $e \in \mathbb{C}^n$ denote the $n \times 1$ vector $(e_1, \ldots, e_n)$, and let

$$\mathrm{circ}(a_0, a_1, \ldots, a_{n-1}) := \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$

denote a circulant matrix. Then, by introducing the permutation matrix $P = \mathrm{circ}(0, 1, 0, \ldots, 0)$, and the matrix $A_1 := P - I = \mathrm{circ}(-1, 1, 0, \ldots, 0)$, we have

$$e = A_1 z.$$

We can rewrite (1) in terms of e as

$$u_i = \frac{1}{2}e_i\left(1 - \frac{b^2}{|e_i|^2}\right) - \frac{1}{2}e_{i-1}\left(1 - \frac{b^2}{|e_{i-1}|^2}\right). \tag{3}$$

We can also write the dynamics $\dot{e}_i = \dot{z}_{i+1} - \dot{z}_i$ as

$$\begin{aligned}\dot{e}_i = \frac{1}{2}e_{i+1}\left(1 - \frac{b^2}{|e_{i+1}|^2}\right) - e_i\left(1 - \frac{b^2}{|e_i|^2}\right) \\ + \frac{1}{2}e_{i-1}\left(1 - \frac{b^2}{|e_{i-1}|^2}\right), \quad i = 1, \ldots, n.\end{aligned} \tag{4}$$

Notice that both (3) and (4) have a singularity if $e_i = 0$ for some i, and thus (3) and (4) are defined on the set

$$\mathcal{S} := \{e \in \mathbb{C}^n : |e_i| > 0, \ \forall i\}.$$

The topology of $\mathcal{S}$ is inherited from the topology of $\mathbb{R}^{2n}$. The system (4) on the set $\mathcal{S}$ can be viewed as a completely separate system from (1). If we impose condition (2) on the e_i's and hence relate the system to (1), then (4) evolves on the set $\mathcal{S}_0 \subset \mathcal{S}$:

$$\mathcal{S}_0 := \{e \in \mathbb{C}^n : |e_i| > 0, \ \forall i, \ \sum_{i=1}^n e_i = 0\}.$$

Note that with the relation $e = A_1 z$, $e \in \mathcal{S}_0$ if and only if $z \in \mathcal{T}$.

We can rewrite the equations (1) and (4) in vector form as follows. First, we introduce the function $\phi : \mathbb{C}\backslash\{0\} \to \mathbb{C}$

$$\phi(s) = \frac{1}{2} s \left(1 - \frac{b^2}{|s|^2}\right). \tag{5}$$

Using this function we can write (3) as

$$u_i = \phi(e_i) - \phi(e_{i-1}). \tag{6}$$

We can extend this function up to vectors by defining $\Phi : \mathcal{S} \to \mathbb{C}^n$ as

$$\Phi(e) = (\phi(e_1), \dots, \phi(e_n)).$$

Noting that $-{A_1}^T = \mathrm{circ}(1, 0, \dots, 0, -1)$ we can write (1) as

$$\dot{z} = -{A_1}^T \Phi(A_1 z) = -{A_1}^T \Phi(e). \tag{7}$$

Finally, writing (4) as

$$\dot{e}_i = \phi(e_{i+1}) - 2\phi(e_i) + \phi(e_{i-1}),$$

and using the fact that

$$-A_1 {A_1}^T = \mathrm{circ}(-2, 1, 0, \dots, 0, 1),$$

we can write the e dynamics as

$$\dot{e} = -A_1 {A_1}^T \Phi(e). \tag{8}$$

Remark 1: In the development of (8) we have taken e_i as a point in the complex plane. However, we can equivalently let e_i be a vector in $\mathbb{R}^2$, and thus $e \in \mathbb{R}^{2n}$. The set $\mathcal{S}$ can be written as $\mathcal{S} = \{e \in \mathbb{R}^{2n} : \|e_i\| > 0, \ \forall i\}$. The function $\phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is then defined as

$$\phi(e_i) = e_i \left(1 - \frac{b^2}{\|e_i\|^2}\right),$$

and $\Phi : \mathcal{S} \to \mathbb{R}^{2n}$ is defined as before. Finally, (8) becomes

$$\dot{e} = -\left(A_1 {A_1}^T \otimes I_2\right) \Phi(e),$$

where $\otimes$ is the Kronecker product and I_2 is the 2×2 identity matrix. ◀

This section proceeds in the following manner. We will study the stability of the system (8) on the set $\mathcal{S}$. From this study we will be able to determine the stability of the system (8) on $\mathcal{S}_0$. This is performed through an application of LaSalle's Theorem. In Lemmas 2 to 9 we will establish the results required to apply LaSalle's Theorem, and in Theorem 11 we state the main result for the system (8). From this result we will be able to infer the stability of system (1) on the set $\mathcal{T}$. This takes place in Theorem 13.

In order to perform a stability analysis of the system (8) on the set $\mathcal{S}$, we need to establish that $\mathcal{S}$ is open and connected. We say that an open and connected set is a *domain* [2].

Lemma 2: The set $\mathcal{S}$ is a domain.

Proof: It is clear that the set $\mathcal{S}$ is open. We will show that $\mathcal{S}$ is path-connected, which implies that $\mathcal{S}$ is connected. Consider a point $e \in \mathcal{S}$. This point consists of n complex numbers $e_1, \dots e_n$, which satisfy $e_i \neq 0$, $\forall i$. That is, no component e_i of $e \in \mathcal{S}$, can lie at the origin of the complex plane. Consider two arbitrary points $p, p' \in \mathcal{S}$. The set $\mathcal{S}$ is path-connected if there exists a function $\sigma(t) : [0,\ 1] \mapsto \mathcal{S}$, such that $\sigma(0) = p$ and $\sigma(1) = p'$. Consider the i^{th} component of p and p':

$$p_i := |p_i| \mathrm{e}^{j\theta_i} \quad \text{and} \quad p'_i := |p'_i| \mathrm{e}^{j\theta'_i}.$$

We would like to find a function $\sigma_i(t) : [0, 1] \mapsto \mathbb{C}/\{0\}$, such that $\sigma_i(0) = p_i$ and $\sigma_i(1) = p'_i$. We can simply let $\sigma_i(t)$ be any smooth function that satisfies the boundary conditions, and does not pass through the origin. For example, $\sigma_i(t)$ could be a function which rotates and scales p_i to p'_i. Hence, letting $\sigma_i(t)$ be any such function, and defining $\sigma(t) = [\sigma_1(t), \dots, \sigma_n(t)]$, we obtain the result that $\mathcal{S}$ is path-connected. This implies that $\mathcal{S}$ is connected. A set which is open and connected is a domain. ■

In order to talk about a solution of the system (8), we must ensure local existence and uniqueness of solutions. A sufficient condition for this is that the right-hand side (RHS) of (8) is locally Lipschitz on $\mathcal{S}$. To show this we must compute the Jacobian of the RHS. This is an instance where the complex representation has its limitations. We will therefore show this using $e \in \mathbb{R}^{2n}$ as developed in Remark 1.

Lemma 3: The right hand side of (8) is locally Lipschitz on $\mathcal{S}$.

Proof: From Remark 1, we can let $e \in \mathbb{R}^{2n}$ and write (8) as $\dot{e} = -\left(A_1 {A_1}^T \otimes I_2\right) \Phi(e) =: f(e)$. From Lemma 3.2 of Khalil [2], f is locally Lipschitz on $\mathcal{S}$ if $f(e)$ and $\partial f / \partial e$ are continuous on $\mathcal{S}$. Letting $e_i = (e_{ix}, e_{iy})$, we can see that the function

$$\phi(e_i) = e_i \left(1 - \frac{b^2}{\|e_i\|^2}\right) = \begin{bmatrix} e_{ix} \\ e_{iy} \end{bmatrix} \left(1 - \frac{b^2}{e_{ix}^2 + e_{iy}^2}\right)$$

is continuous for all $\|e_i\| > 0$ (i.e., on the set $\mathbb{R}^2 \setminus \{0\}$), and thus $f(e)$ is continuous on $\mathcal{S}$. Therefore, it remains to be shown that $\partial f / \partial e$ is continuous on $\mathcal{S}$. We have

$$\frac{\partial f}{\partial e} = -\left(A_1 {A_1}^T \otimes I_2\right) \frac{\partial \Phi}{\partial e}.$$

The matrix $\partial \Phi / \partial e$ is a block diagonal matrix with the 2×2 blocks $\partial \phi(e_i) / \partial e_i$ along the diagonal. By computing the Jacobian $\partial \phi(e_i) / \partial e_i$, it can easily be verified that each block is continuous on $\mathbb{R}^2 \setminus \{0\}$. Therefore, $\partial \Phi / \partial e$ is continuous on $\mathcal{S}$ which implies that $\partial f / \partial e$ is continuous on $\mathcal{S}$. ■

Note that at this point in the development we are not saying that $\mathcal{S}$ is positively invariant with respect to the dynamics (8). Later this will be shown to be true. In the following three lemmas we will establish some properties of the systems (1) and (8).

Lemma 4: Under the dynamics (8), if the trajectory $e(t)$ lies entirely in $\mathcal{S}$, the centroid of $e_1, \dots, e_n$ is stationary. In particular, if a trajectory contained in $\mathcal{S}$ starts in $\mathcal{S}_0$, it remains in $\mathcal{S}_0$ for all time.

Proof: Defining the $n \times 1$ vector of 1's as $\mathbf{1}$, the centroid of $e_1, \dots, e_n$ is given by

$$\tilde{e} := \frac{1}{n} \mathbf{1}^T e.$$

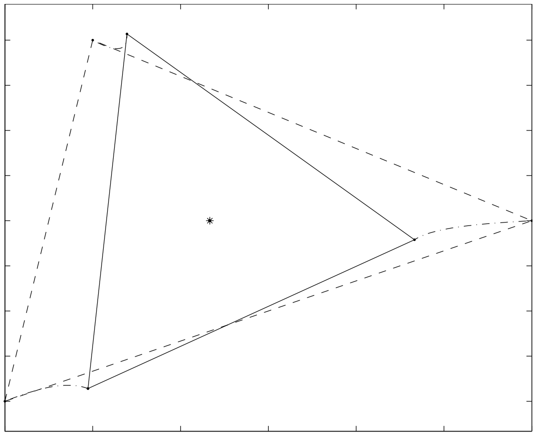


Fig. 1. The evolution of a triangle. The initial triangle is given by the dashed line and the final triangle by the solid line. The stationary centroid is denoted by the $*$.

From (8) we have

$$\dot{e} = -A_1 {A_1}^T \Phi(e).$$

Pre-multiplying this by $\mathbf{1}^T$ we have

$$\begin{aligned} n\dot{\tilde{e}} &= -\mathbf{1}^T A_1 {A_1}^T \Phi(e) = -\mathbf{1}^T (P - I) {A_1}^T \Phi(e) \\ &= -(\mathbf{1}^T P - \mathbf{1}^T) {A_1}^T \Phi(e). \end{aligned}$$

But, $\mathbf{1}$ is an eigenvector of P^T with eigenvalue 1, so $P^T \mathbf{1} = \mathbf{1}$ and thus $\mathbf{1}^T P - \mathbf{1}^T = 0$. Therefore $\dot{\tilde{e}} = 0$, and the centroid of the e_i's is stationary.

Consider a trajectory $e(t) \in \mathcal{S}$, $\forall t \geq 0$. If $e(0) \in \mathcal{S}_0$ then by the definition of $\mathcal{S}_0$, $\mathbf{1}^T e(0) = 0$. Since the centroid is stationary, $\mathbf{1}^T e(t) = 0$ and thus $e(t) \in \mathcal{S}_0$, $\forall t$. ∎

Similarly, the centroid of the z dynamics is stationary.

Lemma 5: Under the dynamics (1), if the trajectory $z(t)$ lies entirely in $\mathcal{T}$, the centroid of $z_1, \ldots, z_n$ is stationary.

Proof: The centroid of the n points is given by $\tilde{z} := \frac{1}{n} \mathbf{1}^T z$. From (7) we have $\dot{z} = -{A_1}^T \Phi(e)$. Pre-multiplying this by $\mathbf{1}^T$ we have

$$\begin{aligned} n\dot{\tilde{z}} &= -\mathbf{1}^T {A_1}^T \Phi(e) = -\mathbf{1}^T (P - I)^T \Phi(e) \\ &= -(\mathbf{1}^T P^T - \mathbf{1}^T) \Phi(e) = 0, \end{aligned}$$

since $\mathbf{1}^T P^T = \mathbf{1}^T$. Therefore $\dot{\tilde{z}} = 0$, and the centroid of the n points is stationary. ∎

In Figure 1 the evolution of a triangle is shown. Notice that the centroid is stationary and the triangle evolves to an equilateral triangle.

Lemma 6: Consider a trajectory $e(t)$ of (8) which lies entirely in $\mathcal{S}$. If the components $e_1, \ldots, e_n$ are collinear at some time t_1, then they are collinear for all $t < t_1$ and $t > t_1$.

Proof: Let $x := \Re\{e\} \in \mathbb{R}^n$. If the points $e_1, \ldots, e_n$ are all collinear at t_1, then we can rotate the coordinate system such that they all lie on the imaginary axis. Then $x(t_1) = 0$. Therefore, defining the function $\psi(x) = x$, and the set

$$\mathcal{L} := \{x \in \mathbb{R}^n : \psi(x) = 0\},$$

we have $x(t_1) \in \mathcal{L}$. Notice that $\partial\psi/\partial x = I_n$, where I_n is the $n \times n$ identity matrix. Therefore, if $L_{\dot{x}}\psi(x(t_1)) = 0$ for all $x(t_1) \in \mathcal{L}$, then $\mathcal{L}$ is an invariant set. We have

$$L_{\dot{x}}\psi(x) = \frac{\partial\psi}{\partial x}\dot{x} = \dot{x}.$$

From (8) we have $\dot{x} = -A_1 {A_1}^T \Re\{\Phi(e)\}$. However, notice that from (5) we can write $\phi(e_i) = e_i k(e_i)$ where

$$k(e_i) := \frac{1}{2}\left(1 - \frac{b^2}{|e_i|^2}\right) \in \mathbb{R}.$$

Therefore, defining $K(e) = \operatorname{diag}(k(e_1), \ldots, k(e_n))$ we have $\Phi(e) = K(e)e$ and thus $\Re\{\Phi(e)\} = K(e)x$. Hence

$$L_{\dot{x}}\psi(x(t_1)) = -A_1 {A_1}^T K(e) x(t_1) = 0,$$

since $x(t_1) = 0$. This implies that $\mathcal{L}$ is an invariant set and if the points are collinear at some time t_1, they are collinear for all time. ∎

Note that this implies that the same collinearity property holds for the z dynamics. In the following lemma we prove two properties of ϕ which will be useful for the upcoming analysis.

Lemma 7: The function $\phi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

$$\phi(s) = \frac{1}{2} s \left(1 - \frac{b^2}{|s|^2}\right),$$

has the following properties:

(i) $\phi(s) = 0$ if and only if $|s| = b$, and

(ii) the restriction of ϕ to $\mathbb{R}^+$ is one-to-one.

Proof: To show (i), we have $\phi(s) = 0$ if and only if $|\phi(s)| = 0$. Hence

$$|\phi(s)| = \frac{1}{2}\left|s\left(1 - \frac{b^2}{|s|^2}\right)\right| = \frac{1}{2}|s|\left|1 - \frac{b^2}{|s|^2}\right| = 0.$$

Since $|s| > 0$ we have

$$|\phi(s)| = 0 \quad \Leftrightarrow \quad 1 - \frac{b^2}{|s|^2} = 0 \quad \Leftrightarrow \quad |s| = b.$$

For (ii), let q be a positive real number. Then

$$\phi(q) = \frac{1}{2}\left(q - \frac{b^2}{q}\right).$$

Taking the derivative of $\phi(q)$ with respect to q we obtain

$$\frac{d\phi}{dq} = \frac{1}{2} + \frac{b^2}{2q^2} > 0, \quad \forall q \in \mathbb{R}^+.$$

Therefore ϕ is monotonically increasing on $\mathbb{R}^+$ which implies that the restriction of ϕ to $\mathbb{R}^+$ is one-to-one. ∎

In general, the function ϕ is not one-to-one. For example, let $s_1 = b/\sqrt{2}$ and $s_2 = -b\sqrt{2}$. Then $\phi(s_1) = \phi(s_2) = -b/\sqrt{2}$.

We will now characterize the equilibria of the system (8) on the set $\mathcal{S}$. To keep the notation compact we introduce the set

$$\mathcal{I} := \{1, 2, \ldots, n\}.$$

We also introduce the unit vector notation

$$\hat{e}_i := \frac{e_i}{|e_i|}. \tag{9}$$

Lemma 8: The equilibria of the system (8) on the set $\mathcal{S}$ are given by

$$\begin{aligned} E :&= \{e \in \mathcal{S} : \Phi(e) \in \ker {A_1}^T\} \\ &= \{e \in \mathcal{S} : \phi(e_i) = \phi(e_j),\ \forall i, j \in \mathcal{I}\}. \end{aligned}$$

Proof: From (8), at equilibrium we have

$$A_1 {A_1}^T \Phi(e) = 0.$$

Pre-multiplying both sides by $\Phi(e)^T$ we have that

$$\Phi(e)^T A_1 A_1^T \Phi(e) = 0 \quad \Rightarrow \quad \|A_1^T \Phi(e)\|^2 = 0.$$

Therefore, in equilibrium, $\Phi(e) \in \ker {A_1}^T$. Since ${A_1}^T = \mathrm{circ}(-1, 0, \ldots, 0, 1)$, this implies that all components of $\Phi(e)$ are equal. ■

Now we will characterize the equilibria of (8) on the set $\mathcal{S}_0 \subset \mathcal{S}$.

Lemma 9: Let $e \in \mathcal{S}_0$ be an equilibrium of (8). If the components $e_1, \ldots, e_n$ are not all collinear, then e lies in the set

$$E_1 := \{e \in \mathcal{S}_0 : |e_i| = b,\ \forall i\}.$$

If the components are collinear, then e lies in

$$E_2 := \left\{ e \in \mathcal{S}_0 : e_i = e_j \ \text{ or } \ e_i = -e_j \frac{b^2}{|e_j|^2},\ \forall i, j \in \mathcal{I} \right\}.$$

Proof: From Lemma 8, at equilibrium all components of $\Phi(e)$ are equal. If $\Phi(e) = 0$, then $\phi(e_i) = 0$, $\forall i$, which from Lemma 7 implies that $|e_i| = b$, $\forall i$.

If $\Phi(e) \neq 0$, then $\phi(e_i)$ must take the same nonzero value in the complex plane for every i. That is,

$$\phi(e_i) = \phi(e_j) \quad \forall i, j \in \mathcal{I}. \tag{10}$$

From (9), we can write this as

$$\hat{e}_i \phi(|e_i|) = \hat{e}_j \phi(|e_j|), \quad \forall i, j \in \mathcal{I}, \tag{11}$$

and so e_i and e_j must be collinear, for all $i, j \in \mathcal{I}$. For simplicity, rotate the coordinate system so that e_i points along the positive real axis. Then we have $\hat{e}_i = 1$ and $\hat{e}_j = \pm 1$, where the sign depends on e_j's orientation relative to e_i. Therefore, from (11) we have $\phi(|e_i|) = \pm\phi(|e_j|)$.

If $\hat{e}_j = 1$ then $\phi(|e_i|) = \phi(|e_j|)$. From Lemma 7, this is satisfied only if $|e_i| = |e_j|$. Combining this with the fact that $\hat{e}_i = \hat{e}_j$ we obtain that (10) is satisfied if $e_i = e_j$.

The other option is that $\hat{e}_j = -1$, in which case $\phi(|e_i|) = -\phi(|e_j|)$, and thus

$$|e_i| - \frac{b^2}{|e_i|} = -\left(|e_j| - \frac{b^2}{|e_j|} \right).$$

Solving this we obtain $|e_i||e_j| = b^2$. Combining this with the fact that $\hat{e}_i = -\hat{e}_j$ we obtain $e_i = -e_j b^2/|e_j|^2$.

So the equilibria fall into two categories. If the points are not all collinear then they lie in the set

$$E_1 := \{e \in \mathcal{S}_0 : |e_i| = b,\ \forall i\}.$$

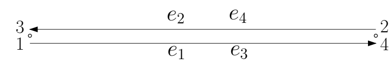
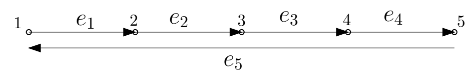
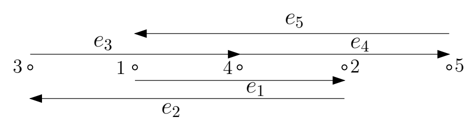


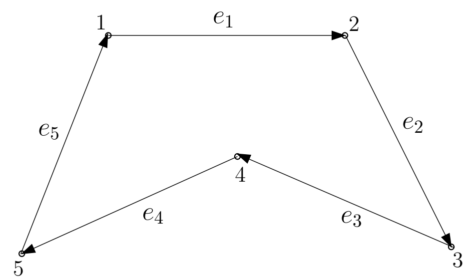
Fig. 2. An equilibrium for $n = 4$ which is in both E_1 and E_2. Here the points are collinear and $|e_i| = b$, $\forall i$.



(a) collinear: $|e_1|, |e_2|, |e_3|, |e_4| = b/\sqrt{6}$, $|e_5| = 5|e_1|$.



(b) collinear: $|e_1|, |e_3|, |e_4| = b(2/3)^{1/2}$, $|e_2|, |e_5| = b(3/2)^{1/2}$



(c) non-collinear: $|e_i| = b$, $\forall i$.

Fig. 3. Example equilibrium formations for $n = 5$ agents.

If they are collinear, they lie in the set

$$E_2 := \left\{ e \in \mathcal{S}_0 : e_i = e_j \ \text{ or } \ e_i = -e_j \frac{b^2}{|e_j|^2},\ \forall i, j \in \mathcal{I} \right\}.$$

■

Notice that if $e \in \mathcal{S}_0$, $e_i = e_j$ cannot be satisfied for all $i, j \in \mathcal{I}$, for if it were then $e_1 = e_2 = \cdots = e_n$, and $\sum_{i=1}^n e_i = ne_1 \neq 0$, which implies that $e \notin \mathcal{S}_0$. Also, the sets E_1 and E_2 are not disjoint if n is even. If an even number of points are in equilibrium and are non-collinear, they must lie in E_1. However, if they are collinear, they can lie in both E_1 and E_2. An example is shown in Figure 2. Figure 3 shows three possible equilibrium formations for $n = 5$ agents on the set $\mathcal{S}_0$. In Figures 3(a) and 3(b), $e \in E_2$, and in Figure 3(c) we have $e \in E_1$.

With these preliminary results in place, we will now introduce two functions which will be used in the application of LaSalle's Theorem. First we introduce the function $g : \mathbb{R}^+ \to \mathbb{R}$:

$$g(q) = \frac{q^2}{2} - b^2 \ln(q) - C, \tag{12}$$

where $C = b^2/2 - b^2 \ln(b)$. Using this function we define the continuously differentiable function $V : \mathcal{S} \to \mathbb{R}$:

$$V(e) := \sum_{i=1}^{n} g(|e_i|). \tag{13}$$

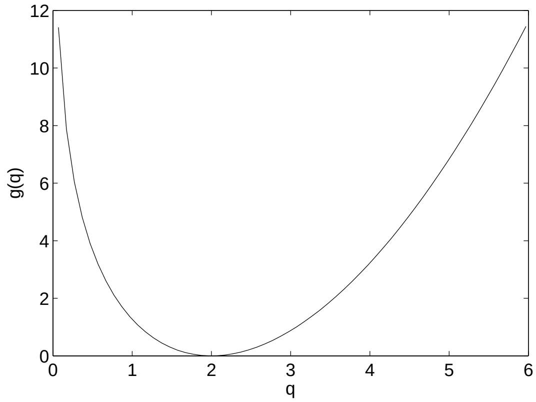


Fig. 4. The function $g(q)$ with $b = 2$. Notice that $g(b) = 0$ is the minimum of the function.

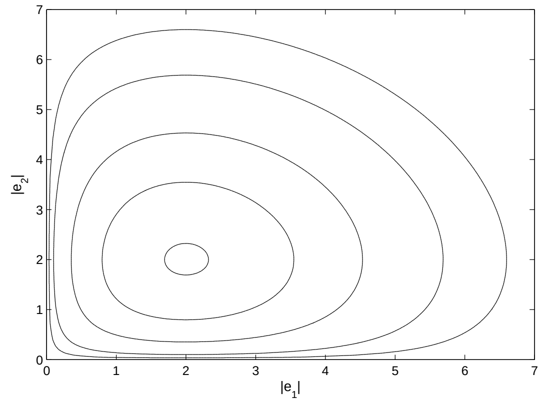


Fig. 5. A few level sets of the function V. In this plot $b = 2$ and so as V decreases, the level sets approach $|e_1| = |e_2| = b$.

Taking the derivative of $g(q)$ with respect to q we obtain:

$$\frac{dg}{dq} = q - \frac{b^2}{q} = 2\phi(q). \tag{14}$$

From Lemma 7 we have that $\phi(q)$ is monotonically increasing and $\phi(q) = 0$ if and only if $q = b$. Therefore $g(q)$ takes its minimum at $g(b) = 0$, as shown in Figure 4. This implies that $V(e) \geq 0$ with $V(e) = 0$ if and only if $|e_i| = b$, $\forall i$. A plot of the level sets of V for $n = 2$ is shown in the $|e_1|, |e_2|$ space in Figure 5.

Lemma 10: If the derivative of $V(e)$ is taken with respect to the dynamics (8), then $\dot{V} \leq 0$ on $\mathcal{S}$, with $\dot{V} = 0$ if and only if $e \in E$ (where E is defined in Lemma 8).

Proof: Taking the Lie derivative of V in (13) we have

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{n} \frac{dg}{de_i}\frac{de_i}{dt} = \sum_{i=1}^{n} \frac{dg}{d|e_i|}\frac{d|e_i|}{de_i}\frac{de_i}{dt} \\ &= \sum_{i=1}^{n} \frac{dg(|e_i|)}{d|e_i|}\frac{d|e_i|}{dt}.\end{aligned}$$

It can be verified that

$$\frac{d}{dt}|e_i| = \frac{d}{dt}\langle e_i, e_i\rangle^{1/2} = \frac{1}{|e_i|}\Re\{\langle e_i, \dot{e}_i\rangle\} = \Re\{\langle \hat{e}_i, \dot{e}_i\rangle\}.$$

From (14) we also have that

$$\frac{dg(|e_i|)}{d|e_i|} = 2\phi(|e_i|).$$

Therefore, we can write $\dot{V}$ as

$$\dot{V} = 2\sum_{i=1}^{n} \phi(|e_i|)\Re\{\langle \hat{e}_i, \dot{e}_i\rangle\} = 2\sum_{i=1}^{n} \Re\{\langle \phi(|e_i|)\hat{e}_i, \dot{e}_i\rangle\}.$$

However, using the fact that $\phi(|e_i|)\hat{e}_i = \phi(e_i)$, we can write this as

$$\dot{V} = 2\sum_{i=1}^{n} \Re\{\langle \phi(e_i), \dot{e}_i\rangle\} = 2\Re\{\langle \Phi(e), \dot{e}\rangle\}.$$

From (8) this becomes

$$\begin{aligned}\dot{V} &= -2\Re\{\langle \Phi(e), A_1{A_1}^T\Phi(e)\rangle\} \\ &= -2\Re\{\Phi(e)^T A_1{A_1}^T\Phi(e)\} = -2\|{A_1}^T\Phi(e)\|^2 \leq 0.\end{aligned}$$

Therefore, $\dot{V} \leq 0$ on $\mathcal{S}$, with equality if and only if $\Phi(e) \in \ker {A_1}^T$. That is, $\dot{V} = 0$ if and only if $e \in E$ (where E is defined in Lemma 8). ■

We say that a trajectory $e(t)$ approaches a set M as $t \to \infty$ if

$$\lim_{t\to\infty} \operatorname{dist}(e(t), M) = 0,$$

where

$$\operatorname{dist}(e(t), M) = \inf_{v\in M} \|e(t) - v\|.$$

Theorem 11: Consider the system (8). For any initial condition $e(0) \in \mathcal{S}$, the solution $e(t)$ approaches E (defined in Lemma 8) as $t \to \infty$. Moreover, for any initial condition $e(0) \in \mathcal{S}_0$, $e(t) \to E_1 \cup E_2$ (defined in Lemma 9) as $t \to \infty$.

Proof: From (12) we have

$$g(|e_i|) = |e_i|^2/2 - b^2\ln(|e_i|) - C,$$

and thus

$$\lim_{|e_i|\to\infty} g(|e_i|) = \infty, \quad \text{and} \quad \lim_{|e_i|\to 0} g(|e_i|) = \infty.$$

Therefore, from the definition of V in (13),

$$\lim_{\|e\|\to\infty} V(e) = \infty,$$

implying that $V(e)$ is radially unbounded, and

$$\lim_{e\to\mathbb{C}^n\setminus\mathcal{S}} V(e) = \infty,$$

implying that $V(e)$ is proper. We define the set

$$\Omega_c = \{e \in \mathbb{C}^n : V(e) \leq c\}, \quad c > 0.$$

Since $V(e)$ is radially unbounded, Ω_c is compact, for all $c > 0$. Also, since $V(e)$ is proper, no level set of $V(e)$ contains a point in $\mathbb{C}^n\setminus\mathcal{S}$ (i.e., no level set of $V(e)$ contains a singularity). Hence, $\Omega_c \subset \mathcal{S}$, for all c. Finally, since $\dot{V} \leq 0$ on $\mathcal{S}$, we have that Ω_c is positively invariant with respect to the dynamics (8).

Therefore, we have a dynamical system (8) which is locally Lipschitz (Lemma 3) on the domain $\mathcal{S}$ (Lemma

2). We have a set $\Omega_c \subset \mathcal{S}$ which is compact and positively invariant with respect to (8). Finally, we have a continuously differentiable function $V : \mathcal{S} \to \mathbb{R}$ such that $\dot{V} \leq 0$ on Ω_c. The set of all points in $\mathcal{S}$ where $\dot{V} = 0$ is given by

$$E = \{e \in \mathcal{S} : \Phi(e) \in \ker {A_1}^T\}.$$

From Lemma 8, E is an invariant set under (8). Therefore, By LaSalle's Theorem (see Theorem 4.4 of [2]), for every initial condition $e(0) \in \Omega_c$, the solution $e(t)$ of (8) approaches $E \cap \Omega_c$ as $t \to \infty$. In addition, for any initial condition $e(0) \in \mathcal{S}$, we can choose c such that $e(0) \in \Omega_c$. Therefore, for every $e(0) \in \mathcal{S}$, $e(t) \to E$ as $t \to \infty$.

If $e(0) \in \mathcal{S}_0$ then by Lemma 4, $e(t) \in \mathcal{S}_0$, $\forall t > 0$. Therefore, $e(t)$ must converge to a point in the set $\mathcal{S}_0 \cap E$ as $t \to \infty$. From Lemma 9, $\mathcal{S}_0 \cap E = E_1 \cup E_2$. Therefore, for every $e(0) \in \mathcal{S}_0$, $e(t) \to E_1 \cup E_2$ as $t \to \infty$. ■

From this theorem we have determined that the sets $\mathcal{S}$ and $\mathcal{S}_0$ are positively invariant under the dynamics (8). Therefore, a trajectory which starts in one of these sets is contained in that set for all time.

Corollary 12: Let $e(t)$ be a trajectory of (8). If $e(0)$ is in $\mathcal{S}_0$, and its components $e_1(0), \ldots, e_n(0)$ are collinear, then $e(t) \to E_2$ as $t \to \infty$.

Proof: From Theorem 11 we have that if $e(0) \in \mathcal{S}_0$, $e(t) \to E_1 \cup E_2$ as $t \to \infty$. From Lemma 9 the collinear equilibria on the set $\mathcal{S}_0$ are given by E_2. Also, from Lemma 6, if $e_1, \ldots, e_n$ are collinear at some time, they are collinear for all time. Therefore, if $e(0) \in \mathcal{S}_0$ and $e_1(0), \ldots, e_n(0)$ are collinear, then $e(t) \to E_2$ as $t \to \infty$. ■

In the previous corollary we have shown that if the e_i's start collinear, then $e(t)$ converges to a collinear equilibrium. Unfortunately, if the points start non-collinear, we have not determined whether they will converge to a collinear or non-collinear equilibrium.

Theorem 13: Let $z(t)$ be a trajectory of (7). If $z(0)$ is in $\mathcal{T}$, then:

(i) $z(t)$ converges to a stationary equilibrium,
(ii) if the components of $z(0)$ are non-collinear, then in the limit as $t \to \infty$, $|z_{i+1} - z_i| = b$, $\forall i$, or the components are collinear,
(iii) if the components of $z(0)$ are collinear, they remain collinear.

Proof: If $z(0) \in \mathcal{T}$, then $e(0) = A_1 z(0) \in \mathcal{S}_0$. From Theorem 11 we have that $e(t) \to E_1 \cup E_2$ as $t \to \infty$. On the set $E_1 \cup E_2$, $\Phi(e) \in \ker {A_1}^T$. From (7) we have that $\dot{z} = -{A_1}^T \Phi(e)$, and so on $E_1 \cup E_2$, $\dot{z} = 0$. Therefore, $z(t)$ converges to a stationary equilibrium.

Since $e(t) \to E_1 \cup E_2$ as $t \to \infty$, either $|e_i| \to b$, $\forall i$, which implies $|z_{i+1} - z_i| \to b$, $\forall i$, or $e(t) \to E_2$ which implies that $z(t)$ converges to a collinear equilibrium.

Finally, if $z_1, \ldots, z_n$ are collinear, then $e_1, \ldots, e_n$ are collinear. By Corollary 12, $e(t)$ converges to a collinear equilibrium point, which implies that $z(t)$ converges to a collinear equilibrium. ■

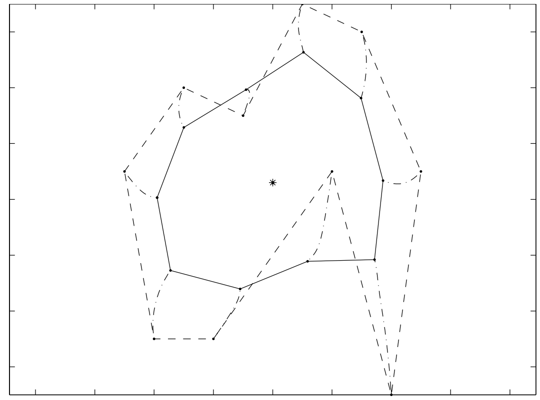


Fig. 6. The evolution of a ten-sided polygon. The initial polygon is given by the dashed line, and the final polygon by the solid line. The length of each side converges to the value b.

In Figure 6 the evolution of a ten sided polygon is shown. The length of each side of the polygon converges to the value b.

IV. Special case of a triangle

In simulation it appears that if the points start non-collinear, they converge to a non-collinear equilibrium. However, through the prior analysis we have not been able to show this for n agents. In this section we will prove it for three agents. To do this we will begin by determining the equilibria of the e dynamics on the set $\mathcal{S}_0$ for three agents.

Lemma 14: For $n = 3$, the collinear equilibria of (8) on the set $\mathcal{S}_0$ are given by $C_1 \cup C_2 \cup C_3$, where

$$C_k := \Big\{e \in \mathbb{C}^3 : |e_k| = b/\sqrt{2},\ e_k = e_{k-1},\\ e_{k+1} = -2e_k\Big\}, \quad k = 1, 2, 3.$$

Proof: From Lemma 6, the collinear equilibria on the set $\mathcal{S}_0$ are given by

$$E_2 := \left\{ e \in \mathcal{S}_0 : e_i = e_j \text{ or } e_i = -e_j \frac{b^2}{|e_j|^2},\ \forall i, j \in \mathcal{I} \right\}$$

With $n = 3$, we have that $e \in E_2$ if $\sum_{i=1}^3 e_i = 0$, and for each i,

(i) $e_i = e_{i-1}$, or
(ii) $e_i = -e_{i-1} \frac{b^2}{|e_{i-1}|^2}$.

To determine the equilibria we introduce the index $k \in \{1, 2, 3\}$. Notice that (i) cannot be satisfied for both $i = k$ and $i = k+1$ for if it were, then $e_{k-1} = e_k = e_{k+1}$ and so either $\sum_{i=1}^3 e_i \neq 0$, or $e_i = 0\ \forall i$, both of which imply that $e \notin \mathcal{S}_0$. Therefore, we have three possible equilibria cases:

Case 1: (i) is satisfied for $i = k$, (ii) is satisfied for $i = k+1$.
Case 2: (i) is satisfied for $i = k+1$, (ii) is satisfied for $i = k$.
Case 3: (ii) is satisfied for $i = k$ and $i = k+1$.

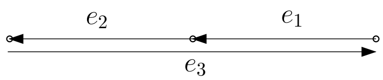


Fig. 7. Three agents in collinear equilibrium with $|e_1| = |e_2| = b/\sqrt{2}$, $|e_3| = 2|e_1|$.

First consider Case 1. From (i) we have $e_k = e_{k-1}$. Substituting this into (2) we have $e_{k+1} = -2e_k$. From (ii) we have

$$e_{k+1} = -e_k \frac{b^2}{|e_k|^2},$$

which, when combined with $e_{k+1} = -2e_k$, gives $|e_k| = b/\sqrt{2}$. Therefore, from Case 1 we obtain

$$|e_k| = \frac{b}{\sqrt{2}}, \quad e_k = e_{k-1}, \quad e_{k+1} = -2e_k.$$

Case 2 is similar. From (i) we have $e_{k+1} = e_k$ and thus from (2), $e_{k-1} = -2e_k$. Setting $i = k$ in (ii) and combining that with $e_{k-1} = -2e_k$ we obtain $|e_k| = b/\sqrt{2}$. Therefore, from Case 2 we obtain

$$|e_k| = \frac{b}{\sqrt{2}}, \quad e_{k+1} = e_k, \quad e_{k-1} = -2e_k.$$

Finally, from Case 3 we have

$$e_k = -e_{k-1}\frac{b^2}{|e_{k-1}|^2} \quad \text{and} \quad e_{k+1} = -e_k\frac{b^2}{|e_k|^2}. \tag{15}$$

From this we have $\hat{e}_k = -\hat{e}_{k-1}$ and $\hat{e}_{k+1} = -\hat{e}_k$ which implies that $\hat{e}_{k-1} = \hat{e}_{k+1}$. Taking the magnitude of the expressions in (15) we obtain $|e_k||e_{k-1}| = b^2 = |e_{k+1}||e_k|$, and thus $|e_{k-1}| = |e_{k+1}|$. Therefore, $e_{k-1} = e_{k+1}$. From (2), we obtain $e_k = -2e_{k-1}$. Combining this with (15) we have $|e_{k-1}| = b/\sqrt{2}$. Therefore, from Case 3 we obtain

$$|e_{k-1}| = \frac{b}{\sqrt{2}}, \quad e_{k+1} = e_{k-1}, \quad e_k = -2e_{k-1}.$$

Notice that the equilibria obtained from the three cases are simply cyclic index shifts of each other. Therefore, the collinear equilibria of the system for $n = 3$ are given by

$$C_k := \Big\{ e \in \mathbb{C}^3 : |e_k| = b/\sqrt{2},\ e_k = e_{k-1}, \\ e_{k+1} = -2e_k \Big\}, \quad k = 1, 2, 3.$$

■

In Figure 7 an equilibrium for $n = 3$ agents is shown. It is interesting to note that $C_k \cap C_{k+1} = \emptyset$, $\forall k$. This can be seen by noting that in equilibrium the magnitudes of the e_i's satisfy

$$|e_{k-1}| = |e_k| = \frac{b}{\sqrt{2}}, \quad |e_{k+1}| = \sqrt{2}b.$$

For $C_k \cap C_{k+1}$ to be nonempty, we require that $b/\sqrt{2} = \sqrt{2}b$, which is satisfied only if $b = 0$.

We would like to show that if the points start non-collinear, the converge to a non-collinear equilibrium point. To do this we require a known result in planar geometry.

Lemma 15: Consider a simple n-sided polygon lying in the complex plane whose vertices, $z_1, \dots, z_n$, are numbered counterclockwise around the polygon. The area inclosed by the polygon is given by

$$A = \frac{1}{2}\sum_{i=1}^{n} \Im\{\langle z_i, z_{i+1}\rangle\}.$$

Because of system (8)'s nonlinear circulant structure, the dynamics of the system are invariant under an index shift. To see this consider the shift $\check{e} := Pe$ where P is the permutation matrix. From (8), we have $\dot{e} = -A_1 {A_1}^T \Phi(e)$. Hence

$$\dot{\check{e}} = P\dot{e} = -PA_1{A_1}^T\Phi(P^{-1}\check{e}).$$

But $\Phi(P^{-1}\check{e}) = (\phi(\check{e}_n), \phi(\check{e}_1), \dots, \phi(\check{e}_{n-1})) = P^{-1}\Phi(\check{e}) = P^T\Phi(\check{e})$, where the last step comes from the fact that $P^{-1} = P^T$. Therefore,

$$\dot{\check{e}} = -PA_1{A_1}^TP^T\Phi(\check{e}) = -(A_1^TP^TPA_1)^T\Phi(\check{e}) \\ = -(A_1^TA_1)^T\Phi(\check{e}) = -A_1{A_1}^T\Phi(\check{e}).$$

Therefore, if $e(t)$ evolves according to (8) then $\check{e}(t)$ also evolves according to (8). Also, notice that if $e \in C_1$ then $\check{e} = Pe \in C_2$ and $P^2\check{e} \in C_3$. Hence, by studying the stability of, say C_2, we are studying the stability of all three collinear equilibrium sets. By exploiting this fact, and using the two previous lemmas, we are now able to prove the main result of this section.

Theorem 16: Let $e(t)$ be a trajectory of (8) starting in $\mathcal{S}_0$ (and thus always lying in $\mathcal{S}_0$). If the components e_1, e_2, e_3 of e start non-collinear, then the components of $\lim_{t\to\infty} e(t)$ are not collinear.

Proof: Since $e(0) \in \mathcal{S}_0$, from Theorem 11, $e(t) \to E_1 \cup E_2$ as $t \to \infty$. For $n = 3$, the collinear equilibria are given by $E_2 := C_1 \cup C_2 \cup C_3$. Since $e_1(0), e_2(0), e_3(0)$ are non-collinear, by Lemma 6, they are non-collinear for all time. Assume by way of contradiction that $e(t) \to C_1 \cup C_2 \cup C_3$ as $t \to \infty$. Because of the circulant structure of (8), this is equivalent to assuming that $e(t) \to C_2$ as $t \to \infty$, where

$$C_2 = \left\{ e \in \mathbb{C}^3 : |e_1| = \frac{b}{\sqrt{2}}, \quad e_2 = e_1, \quad e_3 = -2e_2 \right\}.$$

We can write $e(t) = A_1 z(t)$, where $z(t) \in \mathcal{T}$. Since the e_i's are non-collinear, the z_i's are also non-collinear. Therefore, the z_i's define the vertices of a triangle as shown in Figure 8. We assume without loss of generality that the vertices are initially numbered counterclockwise around the triangle. This implies that that they are numbered counterclockwise for all time; otherwise the vertices would become collinear at some finite time, a contradiction by Lemma 6. From Lemma 15, we can write the area of the triangle as a function of time as

$$A(t) = \frac{1}{2}\sum_{i=1}^{3} \Im\{\langle z_i(t), z_{i+1}(t)\rangle\}.$$

Since we have assumed that $e(t) \to C_2$ as $t \to \infty$, it must also be that $A(t) \to 0$ as $t \to \infty$. Also, since the points are non-collinear for all time, $A(t) > 0$, $\forall t$.

Evaluating the time derivative of A, and using the fact that for $u, v \in \mathbb{C}^n$, $\Im\{\langle u, v\rangle\} = -\Im\{\langle v, u\rangle\}$, we obtain

$$\begin{aligned}\dot{A} &= \frac{1}{2}\sum_{i=1}^{3} \Im\{\langle u_i, z_{i+1}\rangle + \langle z_i, u_{i+1}\rangle\} \\ &= -\frac{1}{2}\sum_{i=1}^{3} \Im\{\langle z_{i+1}, u_i\rangle - \langle z_{i-1}, u_i\rangle\} \\ &= -\frac{1}{2}\sum_{i=1}^{3} \Im\{\langle z_{i+1} - z_{i-1}, u_i\rangle\}.\end{aligned}$$

However, notice that for $n = 3$, $z_{i+1} - z_{i-1} = -e_{i+1}$. Therefore, we have

$$\dot{A} = \frac{1}{2}\sum_{i=1}^{3} \Im\{\langle e_{i+1}, u_i\rangle\}.$$

From (6) we have $u_i = \phi(e_i) - \phi(e_{i-1}) = \phi(|e_i|)\hat{e}_i - \phi(|e_{i-1}|)\hat{e}_{i-1}$. Substituting this in we obtain

$$\dot{A} = \frac{1}{2}\sum_{i=1}^{3} \Im\left\{\phi(|e_i|)\langle e_{i+1}, \hat{e}_i\rangle - \phi(|e_{i-1}|)\langle e_{i+1}, \hat{e}_{i-1}\rangle\right\}.$$

In order to simplify the presentation we let $\phi_i := \phi(|e_i|)$. Introducing this notation we can write $\dot{A}$ as

$$\dot{A} = \frac{1}{2}\sum_{i=1}^{3} \Im\left\{\phi_i|e_{i+1}|\langle \hat{e}_{i+1}, \hat{e}_i\rangle - \phi_{i-1}|e_{i+1}|\langle \hat{e}_{i+1}, \hat{e}_{i-1}\rangle\right\}.$$

Expanding this expression we get

$$\begin{aligned}\dot{A} = \frac{1}{2}\Im\{&\phi_1|e_2|\langle \hat{e}_2, \hat{e}_1\rangle - \phi_3|e_2|\langle \hat{e}_2, \hat{e}_3\rangle + \phi_2|e_3|\langle \hat{e}_3, \hat{e}_2\rangle \\ &- \phi_1|e_3|\langle \hat{e}_3, \hat{e}_1\rangle + \phi_3|e_1|\langle \hat{e}_1, \hat{e}_3\rangle - \phi_2|e_1|\langle \hat{e}_1, \hat{e}_2\rangle\}.\end{aligned}$$

Collecting inner products we obtain

$$\begin{aligned}\dot{A} = &-\frac{1}{2}\Im\{(\phi_1|e_2| + \phi_2|e_1|)\langle \hat{e}_1, \hat{e}_2\rangle \\ &+ (\phi_3|e_2| + \phi_2|e_3|)\langle \hat{e}_2, \hat{e}_3\rangle + (\phi_1|e_3| + \phi_3|e_1|)\langle \hat{e}_3, \hat{e}_1\rangle\}.\end{aligned}$$

Let us define the function:

$$H_i := \Im\{(z_{i-1} - z_i)\overline{(z_{i+1} - z_i)}\} = \rho_{i-1}\rho_i \sin(\beta_i),$$

where β_i is the counterclockwise internal angle from the side $z_i z_{i+1}$ to the side $z_{i-1} z_i$ of a polygon. We can write $\Im\{\langle \hat{e}_{i-1}, \hat{e}_i\rangle\}$ in terms of this function as:

$$\begin{aligned}\Im\{\langle \hat{e}_{i-1}, \hat{e}_i\rangle\} &= \frac{1}{|e_{i-1}||e_i|}\Im\{\overline{(z_i - z_{i-1})}(z_{i+1} - z_i)\} \\ &= -\frac{1}{\rho_{i-1}\rho_i}\Im\{(z_i - z_{i-1})\overline{(z_{i+1} - z_i)}\} \\ &= \frac{1}{\rho_{i-1}\rho_i}\Im\{(z_{i-1} - z_i)\overline{(z_{i+1} - z_i)}\} \\ &= \frac{1}{\rho_{i-1}\rho_i}H_i = \sin(\beta_i).\end{aligned}$$

The angles are shown in Figure 8. Using this, and the fact

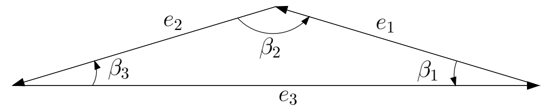


Fig. 8. The triangle for sufficiently large t, showing the three internal angles.

that $\sin(\beta_2) = \sin(\pi - \beta_1 - \beta_3) = \sin(\beta_1 + \beta_3)$ we obtain:

$$\begin{aligned}\dot{A} = &-\frac{1}{2}\left((\phi_1|e_2| + \phi_2|e_1|)\sin(\beta_1 + \beta_3)\right. \\ &\left.+ (\phi_3|e_2| + \phi_2|e_3|)\sin(\beta_3) + (\phi_1|e_3| + \phi_3|e_1|)\sin(\beta_1)\right).\end{aligned}$$

To simplify the following presentation we introduce $\mu := b/\sqrt{2}$. Multiplying $\dot{A}$ by 2, and dividing by $\mu\beta_3 > 0$ we obtain

$$\begin{aligned}\frac{2}{\mu\beta_3}\dot{A} = -\frac{1}{\mu}\Bigg(&(\phi_1|e_2| + \phi_2|e_1|)\frac{\sin(\beta_1 + \beta_3)}{\beta_3} \\ &+ (\phi_3|e_2| + \phi_2|e_3|)\frac{\sin(\beta_3)}{\beta_3} \\ &+ +(\phi_1|e_3| + \phi_3|e_1|)\frac{\sin(\beta_1)}{\beta_3}\Bigg). \quad (16)\end{aligned}$$

As $t \to \infty$, $e(t) \to C_2$. From the definition of C_2 we have that,

$$\lim_{t\to\infty} |e_1| = \mu, \quad \lim_{t\to\infty} |e_2| = \mu, \quad \lim_{t\to\infty} |e_3| = 2\mu. \quad (17)$$

This implies that

$$\begin{aligned}&\lim_{t\to\infty} \phi_1 = \phi(\mu) = -\mu, \quad \lim_{t\to\infty} \phi_2 = \phi(\mu) = -\mu, \\ &\lim_{t\to\infty} \phi_3 = \phi(2\mu) = \mu.\end{aligned} \quad (18)$$

Also from C_2, as $t \to \infty$, $\hat{e}_1(t) \to \hat{e}_2(t) \to -\hat{e}_3(t)$, which implies that $\beta_1(t), \beta_3(t) \to 0$. Finally, since $|e_1(t)| \to |e_2(t)|$ as $t \to \infty$, it follows that the triangle is becoming an isosceles triangle and thus $\beta_1(t) \to \beta_3(t) \to 0$ (that is, β_1 and β_3 approach each other as they approach zero). Therefore, we also have the limits

$$\begin{aligned}&\lim_{t\to\infty} \frac{\sin(\beta_3)}{\beta_3} = 1, \quad \lim_{t\to\infty} \frac{\sin(\beta_1)}{\beta_3} = 1, \\ &\lim_{t\to\infty} \frac{\sin(\beta_1 + \beta_3)}{\beta_3} = 2.\end{aligned} \quad (19)$$

Taking the limit of (16) as $t \to \infty$, and using the expressions in (17), (18), and (19) we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{2}{\mu\beta_3}\dot{A} = -\frac{1}{\mu}\{&(-\mu\mu - \mu\mu)(2) + (\mu\mu - \mu(2\mu))(1) \\ &+ (-\mu(2\mu) + \mu\mu)(1)\} = 6\mu > 0.\end{aligned}$$

This implies that as $t \to \infty$, $\dot{A}(t) \downarrow 0$. Therefore, there exists a time t_1 such that, $\dot{A}(t) > 0, \forall t \geq t_1$. But, $A(t_1) > 0$, and thus

$$A(t) = \int_{t_1}^{t} \dot{A}(s)ds + A(t_1) > A(t_1), \quad \forall t > t_1,$$

a contradiction with our assumption that $e(t) \to C_2$ as $t \to \infty$ (and thus $A(t) \to 0$). Therefore, $e(t)$ does not

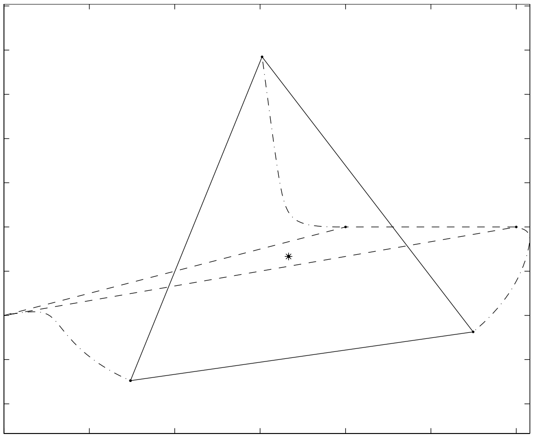


Fig. 9. The evolution of a triangle. The initial triangle is given by the dashed line and the final triangle by the solid line. The stationary centroid is denoted by the $*$.

converge to C_2. This implies that $Pe(t)$ does not converge to C_3 and $P^2e(t)$ does not converge to C_1. Thus, $e(t)$ does not converge to a collinear equilibrium point. ∎

Corollary 17: Let $e(t)$ be a trajectory of (8) starting in $\mathcal{S}_0$. If the components, e_1, e_2, e_3, of e start non-collinear, then $e(t) \to E_1$ as $t \to \infty$.

Proof: From Theorem 11 we know that for every $e(0) \in \mathcal{S}_0$, $e(t)$ converges to the equilibrium set $E_1 \cup E_2$. In Theorem 16 we have shown that $e(t)$ does not converge to the set of collinear equilibria $E_2 = C_1 \cup C_2 \cup C_3$. Therefore $e(t) \to E_1$ as $t \to \infty$. ∎

Theorem 18: Let $z_1(0), z_2(0), z_3(0)$ be distinct, non-collinear points. Under the dynamics of (1), $z_1(t), z_2(t), z_3(t)$ converge to a stationary equilateral triangle with side length equal to b. In addition, their centroid is stationary throughout the evolution.

Proof: Since the points are initially distinct, from Theorem 13, they converge to a stationary equilibrium. Also, since $z \in \mathcal{T}$, this implies that $e = A_1 z \in \mathcal{S}_0$. Since z_1, z_2, z_3 start non-collinear, e_1, e_2, e_3 are also initially non-collinear. Therefore, from Corollary 17, $|e_i| \to b$, $\forall i$ as $t \to \infty$. This implies that $|z_{i+1} - z_i| \to b$, $\forall i$ as $t \to \infty$. Therefore, z_1, z_2, z_3 converge to the vertices of an equilateral triangle, with side length b. From Lemma 4 the centroid of the three points is stationary. ∎

The evolution of a triangle is shown in Figure 9. Even when the vertices start close to being collinear, they converge to an equilateral triangle.

V. Conclusions

In this paper a local control scheme was proposed to stabilize the agents to the vertices of an equilateral polygon. The centroid of the agents is stationary during the evolution. For n agents, we have shown that the agents converge either to the desired formation, or to a collinear equilibrium. In simulation, if the points start non-collinear, they converge to a non-collinear equilibrium. However, this could not be determined from our analysis. For three agents, a full stability analysis was performed. If three agents start distinct and non-collinear, they converge to the vertices of a stationary equilateral triangle, while maintaining a stationary centroid.

References

[1] E. W. Justh and P. S. Krishnaprasad. Equilibria and steering laws for planar formations. *Systems and Control Letters*, 52(1):25–38, 2004.

[2] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[3] Z. Lin, M. Broucke, and B. Francis. Local control strategies for groups of mobile autonomous agents. *IEEE Trans. on Automatic Control*, 49(4):622–629, 2004.

[4] Z. Lin, B. Francis, and M. Maggiore. Necessary and sufficient conditions for formation control of unicycles. *IEEE Trans. on Automatic Control*, 50(1):121–127, 2005.

[5] J. Marshall, M. Broucke, and B. Francis. Formations of vehicles in cyclic pursuit. *IEEE Trans. on Automatic Control*, 49(11):1963–1974, 2004.

[6] R. Olfati-Saber and R. M. Murray. Distributed structural stabilization and tracking for formations of dynamic multi-agents. In *Proc. of the 41st IEEE Conference on Decision and Control*, volume 1, pages 209–215, 2002.

[7] R. Olfati-Saber and R. M. Murray. Graph rigidity and distributed formation stabilization of multi-vehicle systems. In *Proc. of the 41st IEEE Conference on Decision and Control*, volume 3, pages 2965–2971, 2002.

[8] D. Paley, N. E. Leonard, and R. Sepulchre. Collective motion: bistability and trajectory tracking. In *Proc. of the 43rd IEEE Conference on Decision and Control*, volume 2, pages 1932–1937, 2004.

[9] S. L. Smith, M. E. Broucke, and B. A. Francis. Curve shortening and its application to multi-agent systems. In *Proc. of the 41st IEEE Conference on Decision and Control*, pages 2817 – 2822, 2005.

[10] K. Sugihara and I. Suzuki. Distributed motion coordination of multiple robots. In *Proc. of the 5th IEEE International Symposium on Intelligent Control*, pages 138–143, 1990.

[11] H. Tanner, A. Jadbabaie, and G. Pappas. Stable flocking of mobile agents, part i: Fixed topology. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 2010–2015, December 2003.

[12] H. Tanner, A. Jadbabaie, and G. Pappas. Stable flocking of mobile agents, part ii: Dynamic topology. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 2016–2021, December 2003.