

# Dynamic Vehicle Routing for Translating Demands: Stability Analysis and Receding-Horizon Policies

Shaunak D. Bopardikar      Stephen L. Smith      Francesco Bullo      João P. Hespanha

## Abstract

We introduce a problem in which demands arrive stochastically on a line segment, and upon arrival, move with a fixed velocity perpendicular to the segment. We design a receding horizon service policy for a vehicle with speed greater than that of the demands, based on the translational minimum Hamiltonian path (TMHP). We consider Poisson demand arrivals, uniformly distributed along the segment. For a fixed segment width and fixed vehicle speed, the problem is governed by two parameters; the demand speed and the arrival rate. We establish a necessary condition on the arrival rate in terms of the demand speed for the existence of any stabilizing policy. We derive a sufficient condition on the arrival rate in terms of the demand speed that ensures stability of the TMHP-based policy. When the demand speed tends to the vehicle speed, every stabilizing policy must service the demands in the first-come-first-served (FCFS) order; and the TMHP-based policy becomes equivalent to the FCFS policy which minimizes the expected time before a demand is serviced. When the demand speed tends to zero, the sufficient condition on the arrival rate for stability of the TMHP-based policy is within a constant factor of the necessary condition for stability of any policy. Finally, when the arrival rate tends to zero for a fixed demand speed, the TMHP-based policy becomes equivalent to the FCFS policy which minimizes the expected time before a demand is serviced. We numerically validate our analysis and empirically characterize the region in the parameter space for which the TMHP-based policy is stable.

## Index Terms

Dynamic vehicle routing, autonomous vehicles, queueing theory, minimum Hamiltonian path.

## I. INTRODUCTION

Vehicle routing problems are concerned with planning optimal vehicle routes for providing service to a given set of customers. The routes are planned with complete information of the customers, and thus the optimization is static, and typically combinatorial [3]. In contrast, Dynamic Vehicle Routing (DVR) considers scenarios in which not all customer information is known *a priori*, and thus routes must be re-planned as new customer information becomes

Preliminary versions of this work appeared as [1] and as [2]. This material is based upon work supported in part by ARO-MURI Award W911NF-05-1-0219, ONR Award N00014-07-1-0721 and by the Institute for Collaborative Biotechnologies through the grant DAAD19-03-D-0004 from the U.S. Army Research Office.

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available. DVR problems naturally occur in scenarios where autonomous vehicles are deployed in complex and uncertain environments. Examples include search and reconnaissance missions, and environmental monitoring. An early DVR problem is the Dynamic Traveling Repairperson Problem (DTRP) [4], in which customers, or demands arrive sequentially in a region and a service vehicle seeks to serve them by reaching each demand location. In this paper, we introduce a dynamic vehicle routing problem in which the demands move with a specified velocity upon arrival, and we design a novel receding horizon control policy for a single vehicle to service them. This problem has applications in areas such as surveillance and perimeter defense, wherein the demands could be visualized as moving targets trying to cross a region under surveillance by an Unmanned Air Vehicle [5], [6]. Another application is in the automation industry where the demands are objects that arrive continuously on a conveyor belt and a robotic arm performs a pick-and-place operation on them [7].

### *Contributions*

We introduce a dynamic vehicle routing problem in which demands arrive via a stochastic process on a line segment of fixed length, and upon arrival, translate with a fixed velocity perpendicular to the segment. A service vehicle, modeled as a first-order integrator having speed greater than that of the demands, seeks to serve these mobile demands. The goal is to design stable service policies for the vehicle, i.e., the expected time spent by a demand in the environment is finite. We propose a novel receding horizon control policy for the vehicle that services the translating demands as per a translational minimum Hamiltonian path (TMHP).

In this paper, we analyze the problem when the demands are uniformly distributed along the segment and the demand arrival process is Poisson with rate  $\lambda$ . For a fixed length  $W$  of the segment and the vehicle speed normalized to unity, the problem is governed by two parameters; the demand speed  $v$  and the arrival rate  $\lambda$ . Our results are as follows. First, we derive a necessary condition on  $\lambda$  in terms of  $v$  for the existence of a stable service policy. Second, we analyze our novel TMHP-based policy and derive a sufficient condition for  $\lambda$  in terms of  $v$  that ensures stability of the policy. With respect to stability of the problem, we identify two asymptotic regimes: (a) *High speed regime*: when the demands move as fast as the vehicle, i.e.,  $v \rightarrow 1^-$  (and therefore for stability,  $\lambda \rightarrow 0^+$ ); and (b) *High arrival regime*: when the arrival rate tends to infinity, i.e.,  $\lambda \rightarrow +\infty$  (and therefore for stability,  $v \rightarrow 0^+$ ). In the high speed regime, we show that: (i) for existence of a stabilizing policy,  $\lambda$  must converge to zero as  $1/\sqrt{-\ln(1-v)}$ , (ii) every stabilizing policy must service the demands in the first-come-first-served (FCFS) order, and (iii) the TMHP-based policy becomes equivalent to the FCFS policy which we establish is optimal in terms of minimizing the expected time to service a demand. In the high arrival regime, we show that the sufficient condition on  $v$  for the stability of the TMHP-based policy is within a constant factor of the necessary condition on  $v$  for stability of any policy. Third, we identify another asymptotic regime, termed as the *low arrival regime*, in which the arrival rate  $\lambda \rightarrow 0^+$  for a fixed demand speed. In this low arrival regime, we establish that the TMHP-based policy becomes equivalent to the FCFS policy which we establish is optimal in terms of minimizing the expected time to service a demand. Fourth, for the analysis of the TMHP-based policy, we study the classic FCFS policy in which demands are served in the order in which they arrive. We determine necessary and sufficient conditions

on  $\lambda$  for the stability of the FCFS policy. Fifth and finally, we validate our analysis with extensive simulations and provide an empirically accurate characterization of the region in the parameter space of demand speed and arrival rate for which the TMHP-based policy is stable. Our numerical results show that the theoretically established sufficient stability condition for the TMHP-based policy in the high arrival asymptotic regime serves as a very good approximation to the stability boundary, for nearly the entire range of demand speeds.

A plot of the theoretically established necessary and sufficient conditions for stability in the  $v$ - $\lambda$  parameter space is shown in Figure 1. The bottom figures are for the asymptotic regimes of  $\lambda \rightarrow +\infty$ , and  $v \rightarrow 1^-$ , respectively.

### *Related work*

Dynamic vehicle routing (DVR) is an area of research that integrates fundamentals of combinatorics, probability and queueing theories. One of the early versions of a DVR problem is the Dynamic Traveling Repairperson Problem [4] in which the goal is to minimize the expected time spent by each demand before being served. In [4], the authors propose a policy that is optimal in the case of low arrival rate, and several policies within a constant factor of the optimal in the case of high arrival rate. In [8], they consider multiple vehicles, and vehicles with finite service capacity. In [9], a single policy is proposed which is optimal for the case of low arrival rate and performs within a constant factor of the best known policy for the case of high arrival rate. Recently, there has been an upswing in versions of DVR such as in [10] where different classes of demands have been considered; and in [11] which addresses the case of demand impatience. DVR problems addressing motion constraints on the vehicle have been presented in [12], while limited sensing range for the vehicle has been considered in [13]. Problems involving multiple vehicles and with minimal communication have been considered in [14]. In [15], adaptive and decentralized policies are developed for the multiple service vehicle versions, and in [16], a dynamic team-forming variation of the Dynamic Traveling Repairperson problem is addressed. Related dynamic problems include [17], wherein a receding horizon control has been proposed for multiple vehicles to visit target points in uncertain environments; and [18], wherein pickup and delivery problems have been considered.

Another related area of research is the version of the Euclidean Traveling Salesperson Problem with moving points or objects. The static version of the Euclidean Traveling Salesperson Problem consists of determining the minimum length tour through a given set of static points in a region [19]. Vehicle routing with objects moving on straight lines was introduced in [7], in which a fixed number of objects move in the negative  $y$  direction with fixed speed, and the motion of the service vehicle is constrained to be parallel to either the  $x$  or the  $y$  axis. For a version of this problem wherein the vehicle has arbitrary motion, termed as the translational Traveling Salesperson Problem, a polynomial-time approximation scheme has been proposed in [20] to catch all objects in minimum time. Another variation of this problem with object motion on piece-wise straight line paths, and with different but finite object speeds has been addressed in [21]. Other variants of the Euclidean Traveling Salesperson Problem in which the points are allowed to move in different directions have been addressed in [22].

A third area of research related to this work is a geometric location problem such as [23], and [24], where given a set of static demands, the goal is to find supply locations that minimize a cost function of the distance from each

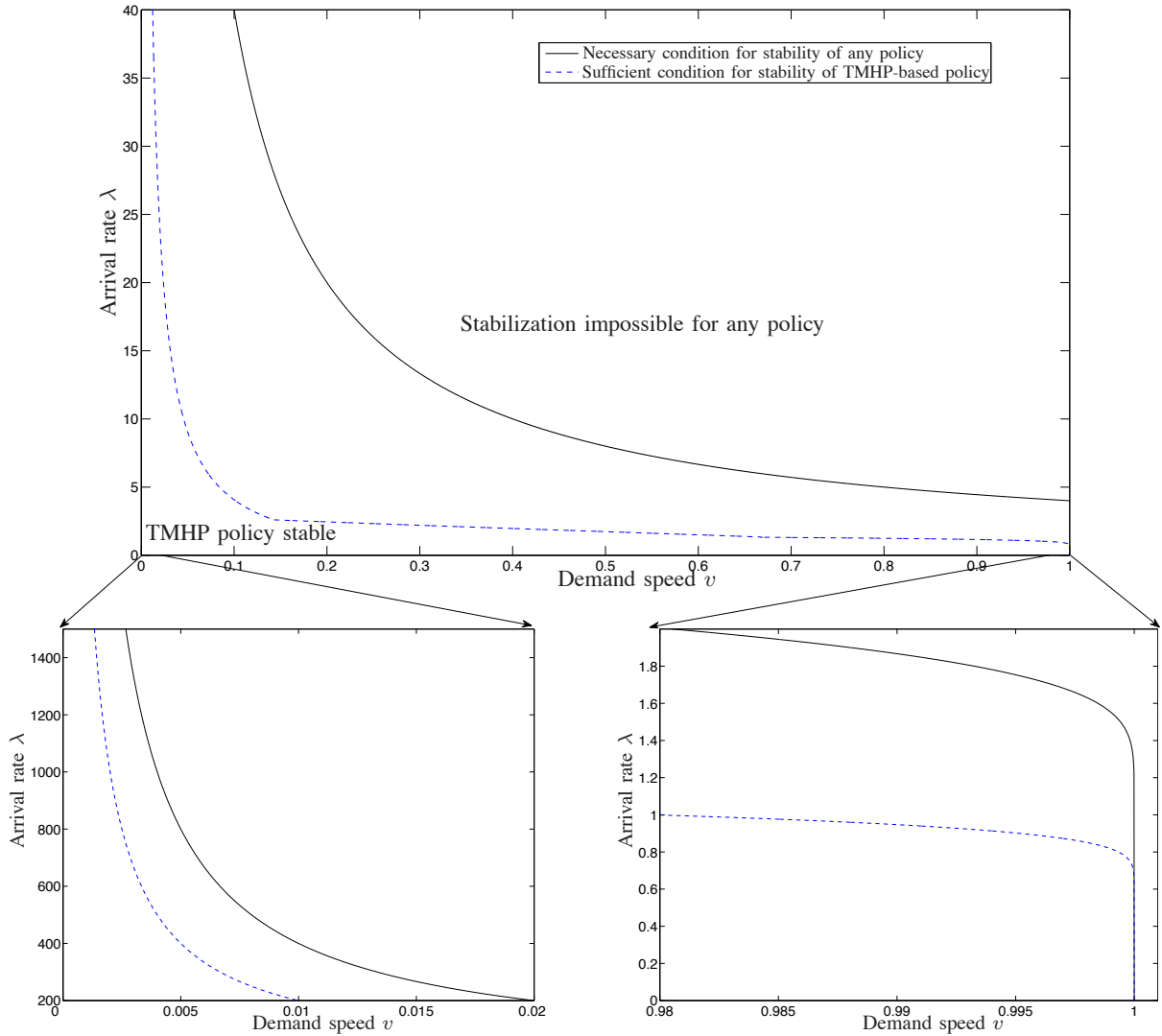


Fig. 1. A summary of stability regions for the TMHP-based policy and the FCFS policy. Stable service policies exist only for the region under the solid black curve. In the top figure, the solid black curve is due to part (i) of Theorem III.1 and the dashed blue curve is due to part (i) of Theorem III.2. In the asymptotic regime shown in the bottom left, the dashed blue curve is described in part (ii) of Theorem III.2, and is different than the one in the top figure. In the asymptotic regime shown in the bottom right, the solid black curve is due to part (ii) of Theorem III.1, and is different from the solid black curve in the top figure.

demand to its nearest supply location. In our problem, in the asymptotic regime of low arrival, when the arrival rate  $\lambda$  tends to zero for a fixed demand speed  $v$ , the problem becomes one of providing optimal coverage. In this regime, the demands are served in a first-come-first-served order; such policies are common in classical queuing theory [25]. Another related work is [26], in which the authors study the problem of deploying robots into a region so as to provide optimal coverage.

Other relevant literature include [27] in which multiple mobile targets are to be kept under surveillance by

multiple mobile sensor agents. The authors in [28] propose mixed-integer-linear-program approach to assigning multiple agents to time-dependent cooperative tasks such as tracking mobile targets.

In an early version of this work [1], [2], we analyzed a preliminary version of the TMHP-based policy and the FCFS policy respectively. In this paper, we present and analyze a unified policy for the problem. We also present new simulation results to numerically determine the region of stability for the policy proposed in this paper.

As in [17] and in [18], the problem addressed in this paper applies several control theoretic concepts to a relevant robotic application. In particular, we design policies or control laws in order to stabilize a stochastic and dynamic system. The solution that we propose is a version of receding horizon control, in which one computes the optimal solution based on all of the present information and then repeatedly re-computes as new information becomes available. Our stability analysis relies on writing the closed-loop dynamical system as a recursion in a state of the system, and then ensuring that this state has a bounded evolution.

### Organization

This paper is organized as follows. A short review of results on optimal motion and combinatorics is presented in Section II. The problem formulation, the TMHP-based policy, and the main results are presented in Section III. The FCFS policy is presented and analyzed in Section IV. Utilizing the results of Section IV, the main results are proven in Section V. Finally, simulation results are presented in Section VI.

## II. PRELIMINARY RESULTS

In this section, we provide some background results which would be used in the remainder of the paper.

### A. Constant bearing control

In this paper, we will use the following result on catching a moving demand in minimum time.

**Definition II.1 (Constant bearing control)** *Given initial locations  $\mathbf{p} := (X, Y) \in \mathbb{R}^2$  and  $\mathbf{q} := (x, y) \in \mathbb{R}^2$  of the service vehicle and a demand, respectively, with the demand moving in the positive  $y$ -direction with constant speed  $v \in ]0, 1[$ , the motion of the service vehicle towards the point  $(x, y + vT)$ , where*

$$T(\mathbf{p}, \mathbf{q}) := \frac{\sqrt{(1-v^2)(X-x)^2 + (Y-y)^2}}{1-v^2} - \frac{v(Y-y)}{1-v^2}, \quad (1)$$

*with unit speed is defined as the constant bearing control.*

Constant bearing control is illustrated in Figure 2 and characterized in the following proposition.

**Proposition II.2 (Minimum time control, [29])** *The constant bearing control is the minimum time control for the service vehicle to reach the moving demand.*

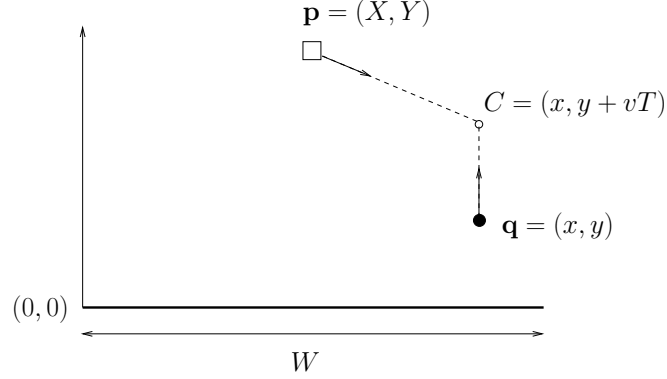


Fig. 2. Constant bearing control. The vehicle moves towards the point  $C := (x, y + vT)$ , where  $x, y, v$  and  $T$  are as per Definition II.1, to reach the demand.

### B. Euclidean and Translational minimum Hamiltonian path (EMHP/TMHP) problems

Given a set of points in the plane, a Euclidean Hamiltonian path is a path that visits each point exactly once. A Euclidean minimum Hamiltonian path (EMHP) is a Euclidean Hamiltonian path that has minimum length. In this paper, we also consider the problem of determining a constrained EMHP which starts at a specified start point, visits a set of points and terminates at a specified end point.

More specifically, the EMHP problem is as follows.

Given  $n$  static points placed in  $\mathbb{R}^2$ , determine the length of the shortest path which visits each point exactly once.

An upper bound on the length of such a path for points in a unit square was given by Few [30]. Here we extend Few's bound to the case of points in a rectangular region. For completeness, we have included the proof in the Appendix.

**Lemma II.3 (EMHP length)** *Given  $n$  points in a  $1 \times h$  rectangle in the plane, where  $h \in \mathbb{R}_{>0}$ , there exists a path that starts from a unit length edge of the rectangle, passes through each of the  $n$  points exactly once, and terminates on the opposite unit length edge, having length upper bounded by*

$$\sqrt{2hn} + h + 5/2.$$

Given a set  $\mathcal{Q}$  of  $n$  points in  $\mathbb{R}^2$ , the Euclidean Traveling Salesperson Problem (ETSP) is to determine the shortest tour, i.e., a closed path that visits each point exactly once. Let  $\text{ETSP}(\mathcal{Q})$  denote the length of the ETSP tour through  $\mathcal{Q}$ . The following is the classic result by Beardwood, Halton, and Hammersly [31].

**Theorem II.4 (Asymptotic ETSP length, [31])** *If a set  $\mathcal{Q}$  of  $n$  points are distributed independently and uniformly in a compact region of area  $A$ , then there exists a constant  $\beta_{\text{TSP}}$  such that, with probability one,*

$$\lim_{n \rightarrow +\infty} \frac{\text{ETSP}(\mathcal{Q})}{\sqrt{n}} = \beta_{\text{TSP}} \sqrt{A}. \quad (2)$$

The constant  $\beta_{\text{TSP}}$  has been estimated numerically as  $\beta_{\text{TSP}} \approx 0.7120 \pm 0.0002$ , [32].

Next, we describe the TMHP problem which was proposed and solved in [20]. This problem is posed as follows.

Given initial coordinates;  $\mathbf{s}$  of a start point,  $\mathcal{Q} := \{\mathbf{q}_1, \dots, \mathbf{q}_n\}$  of a set of points, and  $\mathbf{f}$  of a finish point, all moving with the same constant speed  $v$  and in the same direction, determine a path that starts at time zero from point  $\mathbf{s}$ , visits all points in the set  $\mathcal{Q}$  exactly once and ends at the finish point, and the length  $\mathcal{L}_{T,v}(\mathbf{s}, \mathcal{Q}, \mathbf{f})$  of which is minimum.

In what follows, we wish to determine the TMHP through points which translate in the positive  $y$  direction. We also assume the speed of the service vehicle to be normalized to unity, and hence consider the speed of the points  $v \in ]0, 1[$ . A solution for the TMHP problem is the *Convert-to-EMHP* method:

- (i) For  $v \in ]0, 1[$ , define the conversion map  $\text{cnvrt}_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\text{cnvrt}_v(x, y) = \left( \frac{x}{\sqrt{1-v^2}}, \frac{y}{1-v^2} \right).$$

- (ii) Compute the EMHP that starts at  $\text{cnvrt}_v(\mathbf{s})$ , passes through the set of points given by  $\{\text{cnvrt}_v(\mathbf{q}_1), \dots, \text{cnvrt}_v(\mathbf{q}_n)\}$  and ends at  $\text{cnvrt}_v(\mathbf{f})$ .

- (iii) Move between any two demands using the constant bearing control.

For the Convert-to-EMHP method, the following result is established.

**Lemma II.5 (TMHP length, [20])** *Let the initial coordinates  $\mathbf{s} = (x_s, y_s)$  and  $\mathbf{f} = (x_f, y_f)$ , and the speed of the points  $v \in ]0, 1[$ . The length of the TMHP is*

$$\mathcal{L}_{T,v}(\mathbf{s}, \mathcal{Q}, \mathbf{f}) = \mathcal{L}_E(\text{cnvrt}_v(\mathbf{s}), \{\text{cnvrt}_v(\mathbf{q}_1), \dots, \text{cnvrt}_v(\mathbf{q}_n)\}, \text{cnvrt}_v(\mathbf{f})) + \frac{v(y_f - y_s)}{1 - v^2},$$

where  $\mathcal{L}_E(\text{cnvrt}_v(\mathbf{s}), \{\text{cnvrt}_v(\mathbf{q}_1), \dots, \text{cnvrt}_v(\mathbf{q}_n)\}, \text{cnvrt}_v(\mathbf{f}))$  denotes the length of the EMHP with starting point  $\text{cnvrt}_v(\mathbf{s})$ , passing through the set of points  $\{\text{cnvrt}_v(\mathbf{q}_1), \dots, \text{cnvrt}_v(\mathbf{q}_n)\}$ , and ending at  $\text{cnvrt}_v(\mathbf{f})$ .

This lemma implies the following result: given a start point, a set of points and an end point all of whom translate in the positive vertical direction at speed  $v \in ]0, 1[$ , the order of the points followed by the optimal TMHP solution is the same as the order of the points followed by the optimal EMHP solution through a set of static locations equal to the locations of the moving points at initial time converted via the map  $\text{cnvrt}_v$ .

### III. PROBLEM FORMULATION AND THE TMHP-BASED POLICY

In this section, we pose the dynamic vehicle routing problem with translating demands and present the TMHP-based policy along with the main results.

#### A. Problem Statement

We consider a single service vehicle that seeks to service mobile demands that arrive via a spatio-temporal process on a line segment with length  $W$  along the  $x$ -axis, termed the *generator*. The vehicle is modeled as a first-order integrator with speed upper bounded by one. The demands arrive uniformly distributed on the generator via a

temporal Poisson process with intensity  $\lambda > 0$ , and move with constant speed  $v \in ]0, 1[$  along the positive  $y$ -axis, as shown in Figure 3. We assume that once the vehicle reaches a demand, the demand is served instantaneously. The vehicle is assumed to have unlimited fuel and demand servicing capacity.

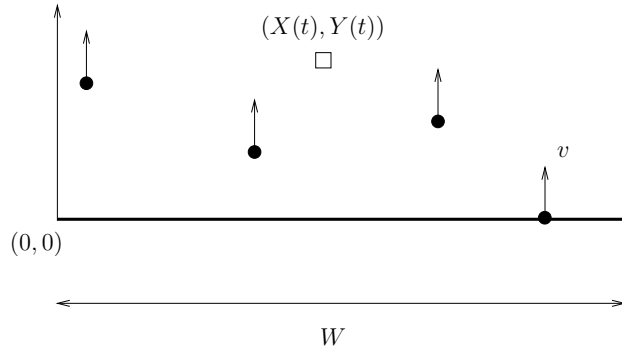


Fig. 3. The problem set-up. The thick line segment is the generator of mobile demands. The dark circle denotes a demand and the square denotes the service vehicle.

We define the environment as  $\mathcal{E} := [0, W] \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$ , and let  $\mathbf{p}(t) = [X(t), Y(t)]^T \in \mathcal{E}$  denote the position of the service vehicle at time  $t$ . Let  $\mathcal{Q}(t) \subset \mathcal{E}$  denote the set of all demand locations at time  $t$ , and  $n(t)$  the cardinality of  $\mathcal{Q}(t)$ . Servicing of a demand  $\mathbf{q}_i \in \mathcal{Q}$  and removing it from the set  $\mathcal{Q}$  occurs when the service vehicle reaches the location of the demand. A static feedback control policy for the system is a map  $\mathcal{P} : \mathcal{E} \times \mathbb{F}(\mathcal{E}) \rightarrow \mathbb{R}^2$ , where  $\mathbb{F}(\mathcal{E})$  is the set of finite subsets of  $\mathcal{E}$ , assigning a commanded velocity to the service vehicle as a function of the current state of the system:  $\dot{\mathbf{p}}(t) = \mathcal{P}(\mathbf{p}(t), \mathcal{Q}(t))$ . Let  $D_i$  denote the time that the  $i$ th demand spends inside the set  $\mathcal{Q}$ , that is, the delay between the generation of the  $i$ th demand and the time it is serviced. The policy  $\mathcal{P}$  is *stable* if under its action,

$$\limsup_{i \rightarrow +\infty} \mathbb{E}[D_i] < +\infty,$$

that is, the steady state expected delay is finite. Equivalently, the policy  $\mathcal{P}$  is stable if under its action,

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[n(t)] < +\infty,$$

that is, if the vehicle is able to service demands at a rate that is—on average—at least as fast as the rate at which new demands arrive. In what follows, our goal is to *design stable control policies* for the system.

To obtain further intuition into stability of a policy, consider the  $v$ - $\lambda$  parameter space. In the asymptotic regime of high speed, where  $v \rightarrow 1^-$ , the arrival rate  $\lambda$  *must* tend to zero for stability, otherwise the service vehicle would have to move successively further away from the generator in expected value, thus making the system unstable. Similarly, stability in the asymptotic region of high arrival, where  $\lambda \rightarrow +\infty$  implies that the demand speed  $v$  *must* tend to zero. Thus, our primary goals are: (i) to characterize regions in the  $v$ - $\lambda$  parameter space in which one can never design *any* stable policy, and (ii) design a novel policy and determine its stability region in the  $v$ - $\lambda$  parameter space, with additional emphasis in the above two asymptotic regimes. In addition, for the asymptotic regime of low



arrival, where for a fixed speed  $v < 1$ , the arrival rate  $\lambda \rightarrow 0^+$ , stability is intuitive as demands arrive very rarely. Hence, in this regime, we seek to minimize the steady state expected delay for a demand.

### B. The TMHP-based policy

We now present a novel receding horizon service policy for the vehicle that is based on the repeated computation of a translational minimum Hamiltonian path through successive groups of outstanding demands. For a given arrival rate  $\lambda$  and demand speed  $v \in ]0, 1[$ , let  $(X^*, Y^*)$  denote the vehicle location in the environment that minimizes the expected time to service a demand once it appears on the generator. The expression for the optimal location  $(X^*, Y^*)$  is postponed to Section IV-A. The TMHP-based policy is summarized in Algorithm 1, and an iteration of the policy is illustrated in Figure 4.

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#### Algorithm 1: The TMHP-based policy

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**Assumes:** The optimal location  $(X^*, Y^*) \in \mathcal{E}$  is given.

- 1 **if** no outstanding demands are present in  $\mathcal{E}$  **then**
  - 2     | Move to the optimal position  $(X^*, Y^*)$ .
  - 3 **else**
  - 4     | Service all outstanding demands by following a translational minimum Hamiltonian path starting from the  
       | vehicle's current location, and terminating at the demand with the lowest  $y$ -coordinate.
  - 5 Repeat.
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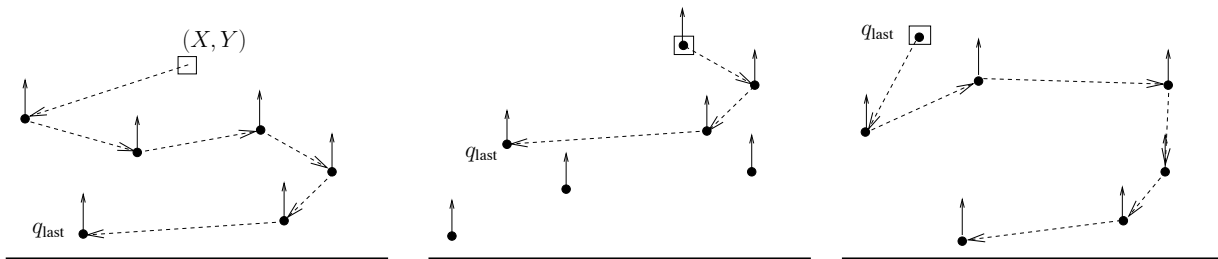


Fig. 4. An iteration of a receding horizon service policy. The vehicle shown as a square serves all outstanding demands shown as black dots as per the TMHP that begins at  $(X, Y)$  and terminates at  $q_{last}$  which is the demand with the least  $y$ -coordinate. The first figure shows a TMHP at the beginning of an iteration. The second figure shows the vehicle servicing the demands through which the TMHP has been computed while new demands arrive in the environment. The third figure shows the vehicle repeating the policy for the set of new demands when it has completed service of the demands present at the previous iteration.

### C. Main Results

The following is a summary of our main results and the locations of their proofs within the paper. We begin with a necessary condition on the problem parameters for stability of any policy, the proof of which is presented

in Section V-A. The condition is policy independent and especially in the asymptotic regime of high speed, i.e.,  $v \rightarrow 1^-$  (and therefore  $\lambda \rightarrow 0^+$ ), we characterize the way in which  $\lambda \rightarrow 0^+$ .

**Theorem III.1 (Necessary condition for stability)** *The following are necessary conditions for the existence of a stabilizing policy:*

(i) For a general  $v \in ]0, 1[$ ,

$$\lambda \leq \frac{4}{vW}.$$

(ii) For the asymptotic regime of high speed, where  $v \rightarrow 1^-$ , every stabilizing policy must serve the demands in the order in which they arrive. Further, the arrival rate must tend to zero as per

$$\lambda \leq \frac{3\sqrt{2}}{W\sqrt{-\ln(1-v)}}.$$

Next, we present a sufficient condition on the problem parameters that ensures stability of the TMHP-based policy, the proof of which is presented in Section V-B. We present a condition for a general value of the demand speed and especially in the asymptotic regime of high arrival, i.e.,  $\lambda \rightarrow +\infty$  (and therefore  $v \rightarrow 0^+$ ), we characterize the way in which  $\lambda \rightarrow +\infty$  and  $v \rightarrow 0^+$ , in order to ensure stability of the TMHP-based policy.

We first introduce the following notation. Let

$$\lambda_{\text{FCFS}}(v, W) := \begin{cases} \frac{3}{W} \sqrt{\frac{1-v}{1+v}}, & \text{for } v \leq v_{\text{suf}}^*, \\ \frac{\sqrt{12v}}{W\sqrt{(1+v)(C_{\text{suf}} - \ln(\frac{1-v}{v}))}}, & \text{otherwise,} \end{cases}$$

where  $C_{\text{suf}} = \pi/2 - \ln(0.5 \cdot \sqrt{3}/\sqrt{2})$ , and  $v_{\text{suf}}^*$  is the solution to  $\sqrt{12v^*} - 3\sqrt{(1-v^*)(C_{\text{suf}} - \ln(1-v^*) + \ln v^*)} = 0$ , and is approximately equal to  $2/3$ .

**Theorem III.2 (Sufficient condition for stability)** *The following are sufficient conditions for stability of the TMHP-based policy.*

(i) For a general  $v \in ]0, 1[$ ,

$$\lambda < \max \left\{ \frac{(1-v^2)^{3/2}}{2vW(1+v)^2}, \lambda_{\text{FCFS}}(v, W) \right\}.$$

(ii) In the asymptotic regime of high arrival where  $\lambda \rightarrow +\infty$  (and so  $v \rightarrow 0^+$ ), the policy is stable if

$$\lambda < \frac{1}{\beta_{\text{TSP}}^2 W v}, \quad \text{where } \beta_{\text{TSP}} \approx 0.7120.$$

A plot of the necessary and sufficient conditions is shown in Fig. 1. In the asymptotic regime of high speed, the sufficient condition from part (i) of Theorem III.2 simplifies to

$$\lambda < \frac{\sqrt{6}}{W\sqrt{-\ln(1-v)}} =: \lambda_{\text{suf}}^{1-},$$

and the necessary condition established in part (ii) of Theorem III.1 simplifies to

$$\lambda \leq \frac{3\sqrt{2}}{W\sqrt{-\ln(1-v)}} =: \lambda_{\text{nec}}^{1-}.$$

In the asymptotic regime of high arrival, the sufficient condition from part (ii) of Theorem III.2 is  $\lambda < 1/(\beta_{\text{TSP}}^2 Wv) =: \lambda_{\text{suf}}^{0+}$ , and the necessary condition established in part (i) of Theorem III.1 is  $\lambda \leq 4/(Wv) =: \lambda_{\text{nec}}^{0+}$ .

Theorems III.1 and III.2 immediately lead to the following corollary.

**Corollary III.3 (Constant factor sufficient condition)** *In the asymptotic regime of*

- (i) *high speed, which is the limit as  $v \rightarrow 1^-$ , the ratio  $\lambda_{\text{nec}}^{1-}/\lambda_{\text{suf}}^{1-} \rightarrow \sqrt{3}$ .*
- (ii) *high arrival, which is the limit as  $\lambda \rightarrow +\infty$ , the ratio  $\lambda_{\text{nec}}^{0+}/\lambda_{\text{suf}}^{0+} \rightarrow 4\beta_{\text{TSP}}^2 \approx 2.027$ .*

The third and final result shows optimality of the TMHP-based policy with respect to minimizing the expected delay in the asymptotic regimes of low arrival and high speed.

**Theorem III.4 (Optimality of TMHP-based policy)** *In the asymptotic regimes of*

- (i) *low arrival, where  $\lambda \rightarrow 0^+$  for a fixed  $v \in ]0, 1[$ , and*
- (ii) *high speed, where  $v \rightarrow 1^-$  (and therefore  $\lambda \rightarrow 0^+$ ),*

*the TMHP-based policy serves the demands in the order in which they arrive, and also minimizes the expected time to service a demand.*

In other words, the TMHP-based policy becomes equivalent to a first-come-first-served (FCFS) policy in the above two asymptotic regimes. Theorem III.4 is proven for the FCFS policy in Section V-B. By the equivalence between FCFS and the TMHP-based policies in the above two asymptotic regimes, the result also holds for the TMHP-based policy.

To study the stability of the TMHP-based policy, we introduce and analyze the FCFS policy in the next section.

#### IV. THE FIRST-COME-FIRST-SERVED (FCFS) POLICY AND ITS ANALYSIS

In this section, we present the FCFS policy and establish some of its properties, and use these properties as tools to analyze the TMHP-based policy.

In the FCFS policy, the service vehicle uses constant bearing control and services the demands in the order in which they arrive. If the environment contains no demands, the vehicle moves to the location  $(X^*, Y^*)$  which minimizes the expected time to catch the next demand to arrive. This policy is summarized in Algorithm 2.

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##### Algorithm 2: The FCFS policy

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**Assumes:** The optimal location  $(X^*, Y^*) \in \mathcal{E}$  is given.

- 1 **if** no outstanding demands are present in  $\mathcal{E}$  **then**
  - 2  $\left| \right.$  Move toward  $(X^*, Y^*)$  until the next demand arrives.
  - 3 **else**
  - 4  $\left[ \right.$  Move using the constant bearing control to service the furthest demand from the generator.
  - 5 Repeat.
-

Figure 5 illustrates an instance of the FCFS policy. The following lemma establishes the relationship between the FCFS policy and the TMHP-based policy.

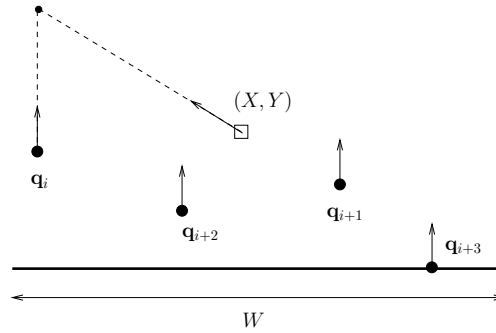


Fig. 5. The FCFS policy. The vehicle services the demands in the order of their arrival in the environment, using the constant bearing control.

**Lemma IV.1 (Relationship between TMHP-based policy and FCFS policy)** *Given an arrival rate  $\lambda$  and a demand speed  $v$ , if the FCFS policy is stable, then the TMHP-based policy is stable.*

*Proof:* Consider an initial vehicle position and a set of outstanding demands, all of which have lower  $y$ -coordinates than the vehicle. Let us compare the amount of time required to service the outstanding demands using the TMHP-based policy with the amount of time required for the FCFS policy. Both policies generate paths through all outstanding demands, starting at the initial vehicle location, and terminating at the outstanding demand with the lowest  $y$ -coordinate. By definition, the TMHP-based policy generates the shortest such path. Thus, the TMHP-based policy will require no more time to service all outstanding demands than the FCFS policy. Since this holds at every iteration of the policy, the region of stability of TMHP-based policy contains the region of stability for the FCFS policy. ■

In the following subsections, we analyze the FCFS policy. We then combine these results with the above lemma to establish analogous results for the TMHP-based policy.

The first question is, how do we compute the optimal position  $(X^*, Y^*)$ ? This will be answered in the following subsection.

#### A. Optimal Vehicle Placement

In this subsection, we study the FCFS policy when  $v \in ]0, 1[$  is fixed and  $\lambda \rightarrow 0^+$ . In this regime, stability is not an issue as demands arrive very rarely, and the problem becomes one of optimally placing the service vehicle (i.e., determining  $(X^*, Y^*)$  in the statement of the FCFS policy).

We seek to place the vehicle at location that minimizes the expected time to service a demand once it appears on the generator. Demands appear at uniformly random positions on the generator and the vehicle uses the constant bearing control to reach the demand. Thus, the expected time to reach a demand generated at position  $\mathbf{q} = (x, 0)$

from vehicle position  $\mathbf{p} = (X, Y)$  is given by

$$\mathbb{E}[T(\mathbf{p}, \mathbf{q})] = \frac{1}{W(1-v^2)} \int_0^W \left( \sqrt{(1-v^2)(X-x)^2 + Y^2} - vY \right) dx. \quad (3)$$

The following lemma characterizes the way in which this expectation varies with the position  $\mathbf{p}$ .

**Lemma IV.2 (Properties of the expected time)** *The expected time  $\mathbf{p} \mapsto \mathbb{E}[T(\mathbf{p}, \mathbf{q})]$  is convex in  $\mathbf{p}$ , for all  $\mathbf{p} \in [0, W] \times \mathbb{R}_{>0}$ . Additionally, there exists a unique point  $\mathbf{p}^* := (W/2, Y^*) \in \mathbb{R}^2$  that minimizes  $\mathbf{p} \mapsto \mathbb{E}[T(\mathbf{p}, \mathbf{q})]$ .*

*Proof:* Regarding the first statement, it suffices to show that the integrand in equation (3),  $T(\mathbf{p}, (x, 0))$  is convex for all  $x$ . To do this we compute the Hessian of  $T((X, Y), (x, 0))$  with respect to  $X$  and  $Y$ . Thus, for  $Y > 0$ ,

$$\begin{bmatrix} \frac{\partial^2 T}{\partial X^2} & \frac{\partial^2 T}{\partial X \partial Y} \\ \frac{\partial^2 T}{\partial Y \partial X} & \frac{\partial^2 T}{\partial Y^2} \end{bmatrix} = \frac{1}{\left( (1-v^2)(X-x)^2 + Y^2 \right)^{3/2}} \begin{bmatrix} Y^2 & Y(X-x) \\ Y(X-x) & (X-x)^2 \end{bmatrix}.$$

The Hessian is positive semi-definite because its determinant is zero and its trace is non-negative. This implies that  $T(\mathbf{p}, \mathbf{q})$  is convex in  $\mathbf{p}$  for each  $\mathbf{q} = (x, 0)$ , from which the first statement follows.

Regarding the second statement, since demands are uniformly randomly generated on the interval  $[0, W]$ , the optimal horizontal position is  $X^* = W/2$ . Thus, it suffices to show that  $Y \mapsto \mathbb{E}[T((W/2, Y), \mathbf{q})]$  is strictly convex. From the  $\partial^2 T / \partial Y^2$  term of the Hessian we see that  $T(\mathbf{p}, \mathbf{q})$  is strictly convex for all  $x \neq W/2$ . But, letting  $\mathbf{p} = (W/2, Y)$  and  $\mathbf{q} = (x, 0)$  we can write

$$\mathbb{E}[T(\mathbf{p}, \mathbf{q})] = \frac{1}{W(1-v^2)} \int_{x \in [0, W] \setminus \{W/2\}} T(\mathbf{p}, \mathbf{q}) dx.$$

The integrand is strictly convex for all  $x \in [0, W] \setminus \{W/2\}$ , implying that  $\mathbb{E}[T(\mathbf{p}, \mathbf{q})]$  is strictly convex on the line  $X = W/2$ , and that the point  $(W/2, Y^*)$  is the unique minimizer.  $\blacksquare$

Lemma IV.2 tells us that there exists a unique point  $\mathbf{p}^* := (X^*, Y^*)$  which minimizes the expected travel time. In addition, we know that  $X^* = W/2$ . Obtaining a closed form expression for  $Y^*$  does not appear to be possible. Computing the integral in equation (3), with  $X = W/2$ , one can obtain

$$\mathbb{E}[T(\mathbf{p}, \mathbf{q})] = \frac{Y}{a} \left( \frac{1}{2} \sqrt{1 + \frac{aW^2}{4Y^2}} - \frac{Y}{\sqrt{aW}} \ln \left( \sqrt{1 + \frac{aW^2}{4Y^2}} - \sqrt{\frac{aW^2}{4Y^2}} \right) - v \right),$$

where  $a = 1 - v^2$ . For each value of  $v$  and  $W$ , this convex expression can be easily numerically minimized over  $Y$ , to obtain  $Y^*$ . A plot of  $Y^*$  as a function of  $v$  for  $W = 1$  is shown in Figure 6.

For the optimal position  $\mathbf{p}^*$ , the expected delay between a demand's arrival and its service completion is

$$D^* := \mathbb{E}[T(\mathbf{p}^*, (x, 0))].$$

Thus, a lower bound on the steady-state expected delay of any policy is  $D^*$ . We now characterize the steady-state expected delay of the FCFS policy  $D_{\text{FCFS}}$ , as  $\lambda$  tends to zero.

**Lemma IV.3 (FCFS optimality)** *Fix any  $v \in ]0, 1[$ . Then in the limit as  $\lambda \rightarrow 0^+$ , the FCFS policy minimizes the expected time to service a demand, i.e.,  $D_{\text{FCFS}} \rightarrow D^*$ .*

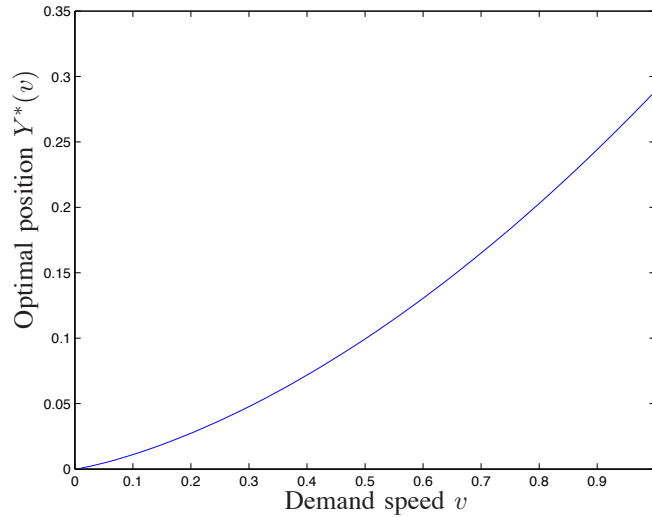


Fig. 6. The optimal position  $Y^*$  of the service vehicle which minimizes the expected distance to a demand, as a function of  $v$ . In this plot, the generator has length  $W = 1$ .

*Proof:* We have shown how to compute the position  $\mathbf{p}^* := (X^*, Y^*)$  which minimizes equation (3). Thus, if the vehicle is located at  $\mathbf{p}^*$ , then the expected time to service the demand is minimized. But, as  $\lambda \rightarrow 0^+$ , the probability that demand  $i + 1$  arrives before the vehicle completes service of demand  $i$  and returns to  $\mathbf{p}^*$  tends to zero. Thus, the FCFS policy is optimal as  $\lambda \rightarrow 0^+$ . ■

**Remark IV.4 (Minimizing the worst-case time)** Another metric that can be used to determine the optimal placement  $(X^*, Y^*)$  is the worst-case time to service a demand. Using an argument identical to that in the proof of Lemma IV.3, it is possible to show that for fixed  $v \in ]0, 1[$ , and as  $\lambda \rightarrow 0^+$ , the FCFS policy, with  $(X^*, Y^*) = (W/2, vW/2)$ , minimizes the worst-case time to service a demand. □

#### B. A Necessary Condition for FCFS Stability

In the previous subsection, we studied the case of fixed  $v$  and  $\lambda \rightarrow 0^+$ . In this subsection, we analyze the problem when  $\lambda > 0$ , and determine necessary conditions on the magnitude of  $\lambda$  that ensure the FCFS policy remains stable. To establish these conditions we utilize a standard result in queueing theory (cf. [25]) which states that a necessary condition for the existence of a stabilizing policy is that  $\lambda \mathbb{E}[T] \leq 1$ , where  $\mathbb{E}[T]$  is the expected time to service a demand (i.e., the travel time between demands). We begin with the following result.

**Proposition IV.5 (Special case of equal speeds)** *For  $v = 1$ , there does not exist a stabilizing policy.*

*Proof:* When  $v = 1$ , each demand and the service vehicle move at the same speed. If a demand has a higher vertical position than the service vehicle, then clearly the service vehicle cannot reach it. The same impossibility

result holds if the demand has the same vertical position and a distinct horizontal position as the service vehicle. In summary, a demand can be reached only if the service vehicle is above the demand. Next, note that the only policy that ensures that a demand's  $y$ -coordinate never exceeds that of the service vehicle (i.e., that all demands remain below the service vehicle at all time) is the FCFS policy. In what follows, we prove the proposition statement by computing the expected time to travel between demands using the FCFS policy. First, consider a vehicle location  $\mathbf{p} := (X, Y)$  and a demand location with initial location  $\mathbf{q} := (x, y)$ , the minimum time  $T$  in which the vehicle can reach the demand is given by

$$T(\mathbf{p}, \mathbf{q}) = \frac{(X - x)^2 + (Y - y)^2}{2(Y - y)}, \quad \text{if } Y > y, \quad (4)$$

and is undefined if  $Y \leq y$ . Second, assume there are many outstanding demands below the service vehicle, and none above. Suppose the service vehicle completed the service of demand  $i$  at time  $t_i$  and position  $(x_i(t_i), y_i(t_i))$ . Let us compute the expected time to reach demand  $i + 1$ , with location  $(x_{i+1}(t_i), y_{i+1}(t_i))$ . Since arrivals are Poisson, it follows that  $y_i(t_i) > y_{i+1}(t_i)$ , with probability one. To simplify notation, we define  $\Delta x = |x_i(t_i) - x_{i+1}(t_i)|$  and  $\Delta y = y_i(t_i) - y_{i+1}(t_i)$ . Then, from equation (4)

$$T(\mathbf{q}_i, \mathbf{q}_{i+1}) = \frac{\Delta x^2 + \Delta y^2}{2\Delta y} = \frac{1}{2} \left( \frac{\Delta x^2}{\Delta y} + \Delta y \right).$$

Taking expectation and noting that  $\Delta x$  and  $\Delta y$  are independent,

$$\mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] = \frac{1}{2} \left( \mathbb{E}[\Delta x^2] \mathbb{E}\left[\frac{1}{\Delta y}\right] + \mathbb{E}[\Delta y] \right).$$

Now, we note that  $\mathbb{E}[\Delta y] = 1/\lambda$ , that  $\mathbb{E}[\Delta x^2]$  is a positive constant independent of  $\lambda$ , and that

$$\mathbb{E}\left[\frac{1}{\Delta y}\right] = \int_{y=0}^{+\infty} \frac{1}{y} \lambda e^{-\lambda y} dy = +\infty.$$

Thus  $\mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] = +\infty$ , and for every  $\lambda > 0$ ,

$$\lambda \mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] = +\infty.$$

This means that the necessary condition for stability, i.e.,  $\lambda \mathbb{E}[T(\mathbf{q}_i, \mathbf{q}_{i+1})] \leq 1$ , is violated. Thus, there does not exist a stabilizing policy. ■

Next we look at the FCFS policy and give a necessary condition for its stability.

**Lemma IV.6 (Necessary stability condition for FCFS)** *A necessary condition for the stability of the FCFS policy is*

$$\lambda \leq \begin{cases} \frac{3}{W}, & \text{for } v \leq v_{\text{nec}}^*, \\ \frac{3\sqrt{2v}}{W \sqrt{(1+v) \left( C_{\text{nec}} - \ln\left(\frac{\sqrt{1-v^2}}{v}\right)\right)}}, & \text{otherwise,} \end{cases}$$

where  $C_{\text{nec}} = 0.5 + \ln(2) - \gamma$ , where  $\gamma$  is the Euler constant; and  $v_{\text{nec}}^*$  is the solution to the equation  $2v - (1+v)(C_{\text{nec}} - 0.5 \cdot \ln(1-v^2) + \ln v) = 0$ , and is approximately equal to 4/5.

*Proof:* Suppose the service vehicle completed the service of demand  $i$  at time  $t_i$  at position  $(x_i(t_i), y_i(t_i))$ , and demand  $i + 1$  is located at  $(x_{i+1}(t_i), y_{i+1}(t_i))$ . Define  $\Delta x := |x_i(t_i) - x_{i+1}(t_i)|$  and  $\Delta y := y_i(t_i) - y_{i+1}(t_i)$ . For  $v \in ]0, 1[$ , the travel time between demands is given by

$$T = \frac{1}{1-v^2} \left( \sqrt{(1-v^2)\Delta x^2 + \Delta y^2} - v\Delta y \right). \quad (5)$$

Observe that the function  $T$  is convex in  $\Delta x$  and  $\Delta y$ . Jensen's inequality leads to

$$\mathbb{E}[T] \geq \frac{1}{1-v^2} \left( \sqrt{(1-v^2)(\mathbb{E}[\Delta x])^2 + (\mathbb{E}[\Delta y])^2} - v\mathbb{E}[\Delta y] \right).$$

Substituting the expressions for the expected values, we obtain

$$\mathbb{E}[T] \geq \frac{1}{1-v^2} \left( \sqrt{(1-v^2)\frac{W^2}{9} + \frac{v^2}{\lambda^2} - \frac{v^2}{\lambda}} \right).$$

From the necessary condition for stability, we must have

$$\lambda \mathbb{E}[T] \leq 1 \quad \iff \quad \lambda \frac{1}{1-v^2} \left( \sqrt{\frac{(1-v^2)W^2}{9} + \frac{v^2}{\lambda^2} - \frac{v^2}{\lambda}} \right) \leq 1.$$

By simplifying, we obtain

$$\lambda \leq \frac{3}{W}. \quad (6)$$

This provides a good necessary condition for low  $v$ . Next, we obtain a much better necessary condition for large  $v$ .

Since  $T$  is convex in  $\Delta x$ , we apply Jensen's inequality to equation (5) to obtain

$$\mathbb{E}[T|\Delta y] \geq \frac{1}{1-v^2} \left( \sqrt{(1-v^2)W^2/9 + \Delta y^2} - v\Delta y \right), \quad (7)$$

where  $\mathbb{E}[\Delta x] = W/3$ . Now, the random variable  $\Delta y$  is distributed exponentially with parameter  $\lambda/v$  and probability density function

$$f(y) = \frac{\lambda}{v} e^{-\lambda y/v}.$$

Un-conditioning equation (7) on  $\Delta y$ , we obtain

$$\mathbb{E}[T] = \int_0^{+\infty} \mathbb{E}[T|y]f(y)dy \geq \frac{\lambda}{v(1-v^2)} \int_0^{+\infty} \left( \sqrt{\frac{(1-v^2)W^2}{9} + y^2} - vy \right) e^{-\lambda y/v} dy. \quad (8)$$

The right hand side can be evaluated using the software Maple<sup>®</sup> and equals

$$\frac{\pi W}{2 \cdot 3\sqrt{1-v^2}} \left[ \mathbf{H}_1 \left( \frac{\lambda W \sqrt{1-v^2}}{3v} \right) - \mathbf{Y}_1 \left( \frac{\lambda W \sqrt{1-v^2}}{3v} \right) \right] - \frac{v^2}{\lambda(1-v^2)},$$

where  $\mathbf{H}_1 : \mathbb{R} \rightarrow \mathbb{R}$  is the 1st order Struve function and  $\mathbf{Y}_1 : \mathbb{R} \rightarrow \mathbb{R}$  is 1st order Bessel function of the 2nd kind [33]. Using a Taylor series expansion of the function  $\mathbf{H}_1(z) - \mathbf{Y}_1(z)$  about  $z = 0$ , followed by a subsequent analysis of the higher order terms, one can show that

$$\mathbf{H}_1(z) - \mathbf{Y}_1(z) \geq \frac{1}{\pi} \left( \frac{2}{z} + C_{\text{nec}} z - z \ln(z) \right), \quad \forall z \geq 0,$$



where  $C_{\text{nec}} = 1/2 + \ln(2) - \gamma$ , and  $\gamma$  is the Euler constant. This inequality implies that equation (8) can be written as

$$\mathbb{E}[T] \geq \frac{v}{\lambda(1+v)} + \frac{\lambda W}{18v} \left( C_{\text{nec}} - \ln \left( \frac{\lambda W \sqrt{1-v^2}}{3v} \right) \right),$$

where we have used the fact that

$$\frac{v}{\lambda(1-v)^2} - \frac{v^2}{\lambda(1-v^2)} = \frac{v}{\lambda(1+v)}.$$

To obtain a stability condition on  $\lambda$  we wish to remove  $\lambda$  from the  $\ln$  term. To do this, note that from equation (6) we have  $\lambda W/3 < 1$ , and thus

$$\mathbb{E}[T] \geq \frac{v}{\lambda(1+v)} + \frac{\lambda W}{18v} \left( C_{\text{nec}} - \ln \frac{W\lambda}{3} - \ln \frac{W\sqrt{1-v^2}}{3v} \right) \geq \frac{v}{\lambda(1+v)} + \frac{\lambda W}{18v} \left( C_{\text{nec}} - \ln \left( \frac{\sqrt{1-v^2}}{v} \right) \right).$$

The necessary stability condition is  $\lambda \mathbb{E}[T] \leq 1$ , from which a necessary condition for stability is

$$\frac{\lambda^2 W}{18v} \left( C_{\text{nec}} - \ln \left( \frac{\sqrt{1-v^2}}{v} \right) \right) \leq 1 - \frac{v}{1+v} = \frac{1}{1+v}.$$

Solving for  $\lambda$  when  $\ln(\sqrt{1-v^2}/v) < C_{\text{nec}}$ , we obtain that

$$\lambda \leq \frac{3\sqrt{2v}}{W\sqrt{(1+v)\left(C_{\text{nec}} - \ln\left(\frac{\sqrt{1-v^2}}{v}\right)\right)}}, \quad (9)$$

The condition  $C_{\text{nec}} > \ln(\sqrt{1-v^2}/v)$ , implies that the above bound holds for all  $v > 1/\sqrt{1+e^{2C_{\text{nec}}}}$ . We now have two bounds; equation (6) which holds for all  $v \in ]0, 1[$ , and equation (9) which holds for  $v > 1/2$ . The final step is to determine the values of  $v$  for which each bound is active. To do this, we set the right-hand side of equation (6) equal to the right-hand side of equation (9) and denote the solution by  $v_{\text{nec}}^*$ . Thus, the necessary condition for stability is given by equation (6) when  $v \leq v_{\text{nec}}^*$ , and by equation (9) when  $v > v_{\text{nec}}^*$ . ■

### C. A Sufficient Condition for FCFS stability

In Section IV-B, we determined a necessary condition for stability of the FCFS policy. In this subsection, we will derive the following sufficient condition on the arrival rate that ensures stability for the FCFS policy. To establish this condition, we utilize a standard result in queueing theory (cf. [25]) which states that a sufficient condition for the existence of a stabilizing policy is that  $\lambda \mathbb{E}[T] < 1$ , where  $\mathbb{E}[T]$  is the expected time to service a demand (i.e., the travel time between demands).

**Lemma IV.7 (Sufficient stability condition for FCFS)** *The FCFS policy is stable if*

$$\lambda < \begin{cases} \frac{3}{W} \sqrt{\frac{1-v}{1+v}}, & \text{for } v \leq v_{\text{suf}}^*, \\ \frac{\sqrt{12v}}{W\sqrt{(1+v)\left(C_{\text{suf}} - \ln\left(\frac{1-v}{v}\right)\right)}}, & \text{otherwise,} \end{cases}$$

where  $C_{\text{suf}} = \pi/2 - \ln(0.5 \cdot \sqrt{3}/\sqrt{2})$ , and  $v_{\text{suf}}^*$  is the solution to  $\sqrt{12v^*} - 3\sqrt{(1-v^*)(C_{\text{suf}} - \ln(1-v^*) + \ln v^*)} = 0$ , and is approximately equal to 2/3.

*Proof:* We begin with the expression for the travel time between two consecutive demands using the constant bearing control (cf. Definition II.1).

$$T = \frac{\sqrt{(1-v^2)\Delta x^2 + \Delta y^2}}{1-v^2} - \frac{v\Delta y}{1-v^2} \leq \frac{|\Delta x|}{\sqrt{1-v^2}} + \frac{\Delta y}{1-v^2}, \quad (10)$$

where we used the inequality  $\sqrt{a^2 + b^2} \leq |a| + |b|$ . Taking expectation,

$$\mathbb{E}[T] \leq \frac{W}{3\sqrt{1-v^2}} + \frac{v}{\lambda(1-v^2)},$$

since the demands are distributed uniformly in the  $x$ -direction and Poisson in the  $y$ -direction. A sufficient condition for stability is

$$\lambda \mathbb{E}[T] < 1 \quad \iff \quad \lambda < \frac{3}{W} \sqrt{\frac{1-v}{1+v}}. \quad (11)$$

The upper bound on  $T$  given by equation (10) is very conservative except for the case when  $v$  is very small. Alternatively, taking expected value of  $T$  conditioned on  $\Delta y$ , and applying Jensen's inequality to the square-root part, we obtain

$$\mathbb{E}[T|\Delta y] \leq \frac{1}{1-v^2} \left( \sqrt{(1-v^2)W^2/6 + \Delta y^2} - v\Delta y \right),$$

since  $\mathbb{E}[\Delta x^2] = W^2/6$ . Following steps which are similar to those between equation (7) and equation (8), we obtain

$$\mathbb{E}[T] \leq \frac{\pi W}{2 \cdot \sqrt{6}\sqrt{1-v^2}} \left[ \mathbf{H}_1 \left( \frac{\lambda W \sqrt{1-v^2}}{\sqrt{6}v} \right) - \mathbf{Y}_1 \left( \frac{\lambda W \sqrt{1-v^2}}{\sqrt{6}v} \right) \right] - \frac{v^2}{\lambda(1-v^2)}. \quad (12)$$

In [33], polynomial approximations have been provided for the Struve and Bessel functions in the intervals  $[0, 3]$  and  $[3, +\infty)$ . We seek an upper bound for the right-hand side of (12) when  $v$  is sufficiently large, i.e., when the argument of  $\mathbf{H}_1$  and  $\mathbf{Y}_1$  is small. From [33], we know that

$$\mathbf{H}_1(z) \leq \frac{z}{2}, \quad \mathbf{Y}_1(z) \geq \frac{2}{\pi} \left( \mathbf{J}_1(z) \ln \frac{z}{2} - \frac{1}{z} \right), \quad \text{and } \mathbf{J}_1(z) \leq \frac{z}{2}, \quad \text{for } 0 \leq z \leq 3,$$

where  $z := \lambda W \sqrt{1-v^2}/(\sqrt{6}v)$ , and  $\mathbf{J}_1 : \mathbb{R} \rightarrow \mathbb{R}$  denotes the Bessel function of the first kind. To obtain a lower bound on  $\mathbf{Y}_1(z)$ , we observe that if  $0 \leq z \leq 2$ , then due to the  $\ln$  term in the above lower bound for  $\mathbf{Y}_1(z)$ , we can substitute  $z/2$  in place of  $\mathbf{J}_1(z)$ . Thus, we obtain

$$\mathbf{H}_1(z) \leq \frac{z}{2}, \quad \mathbf{Y}_1(z) \geq \frac{2}{\pi} \left( \frac{z}{2} \ln \frac{z}{2} - \frac{1}{z} \right), \quad \text{for } 0 \leq z \leq 2. \quad (13)$$

Substituting into equation (12), we obtain

$$\mathbb{E}[T] \leq \frac{\pi W}{2 \cdot \sqrt{6}\sqrt{1-v^2}} \left[ \frac{\lambda W \sqrt{1-v^2}}{2\sqrt{6}v} + \frac{2}{\pi} \left( \frac{\sqrt{6}v}{\lambda W \sqrt{1-v^2}} - \frac{\lambda W \sqrt{1-v^2}}{2\sqrt{6}v} \ln \frac{\lambda W \sqrt{1-v^2}}{2\sqrt{6}v} \right) \right] - \frac{v^2}{\lambda(1-v^2)},$$

which yields

$$\mathbb{E}[T] \leq \frac{\lambda W^2}{12v} \left( \frac{\pi}{2} - \ln \frac{\lambda W}{3} - \ln \frac{\sqrt{3}\sqrt{1-v^2}}{2\sqrt{2}v} \right) - \frac{1}{\lambda(1+v)}. \quad (14)$$

Now, let  $\lambda^*$  be the least upper bound on  $\lambda$  for which the FCFS policy is unstable, i.e., for every  $\lambda < \lambda^*$ , the FCFS policy is stable. To obtain  $\lambda^*$ , we need to solve  $\lambda^* \mathbb{E}[T] = 1$ . Using equation (14), we can obtain a lower

bound on  $\lambda^*$  by simplifying

$$\frac{\lambda^{*2}W^2}{12v} \left( \frac{\pi}{2} - \ln \frac{\lambda^*W}{3} - \ln \frac{\sqrt{3}\sqrt{1-v^2}}{2\sqrt{2}v} \right) - \frac{1}{1+v} \geq 1.$$

From the condition given by equation (11), the second term in the parentheses satisfies

$$\frac{\lambda^*W}{3} > \sqrt{\frac{1-v}{1+v}}.$$

Thus, we obtain,

$$\lambda^* \geq \frac{\sqrt{12v}}{W\sqrt{(1+v)(C_{\text{suf}} - \ln(\frac{1-v}{v}))}},$$

where  $C_{\text{suf}} = \pi/2 - \ln(0.5 \cdot \sqrt{3}/\sqrt{2})$ . Since  $\lambda < \lambda^*$  implies stability, a sufficient condition for stability is

$$\lambda < \frac{\sqrt{12v}}{W\sqrt{(1+v)(C_{\text{suf}} - \ln(\frac{1-v}{v}))}}. \quad (15)$$

To determine the value of the speed  $v_{\text{suf}}^*$  beyond which this is a less conservative condition than equation (11), we solve

$$\frac{\sqrt{12v_{\text{suf}}^*}}{W\sqrt{(1+v_{\text{suf}}^*)(C_{\text{suf}} - \ln(\frac{1-v_{\text{suf}}^*}{v_{\text{suf}}^*}))}} = \frac{3}{W}\sqrt{\frac{1-v_{\text{suf}}^*}{1+v_{\text{suf}}^*}}.$$

For  $v > v_{\text{suf}}^*$ , one can verify that the numerical value of the argument of the Struve and Bessel functions is less than 2, and so the bounds given by equation (13) used in this analysis are valid. Thus, a sufficient condition for stability is given by equation (11) for  $v \leq v_{\text{suf}}^*$ , and by equation (15) for  $v > v_{\text{suf}}^*$ . ■

**Remark IV.8 (Tightness in limiting regimes)** As  $v \rightarrow 0^+$ , the sufficient condition for FCFS stability becomes  $\lambda < 3/W$ , which is exactly equal to the necessary condition given in Lemma IV.6. Thus, the condition for stability is asymptotically tight in this limiting regime.

Figure 7 shows a comparison of the necessary and sufficient stability conditions for the FCFS policy. It should be noted that  $\lambda$  converges to  $0^+$  extremely slowly as  $v$  tends to  $1^-$ , and still satisfy the sufficient stability condition in Lemma IV.7. For example, with  $v = 1 - 10^{-6}$ , the FCFS policy can stabilize the system for an arrival rate of  $3/(5W)$ . □

## V. PROOFS OF THE MAIN RESULTS

In this section, we present the proofs of the main results which were presented in Section III-C.

### A. Proof of Theorem III.1

We first present the proof of part (i). We begin by looking at the distribution of demands in the service region.

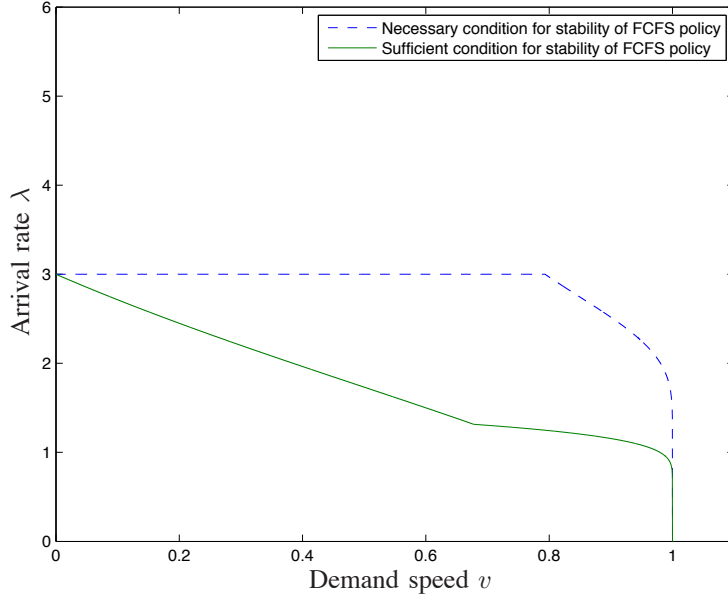


Fig. 7. The necessary and sufficient conditions for the stability for the FCFS policy. The dashed curve is the necessary condition for stability as established in Lemma IV.6; while the solid curve is the sufficient condition for stability as established in Lemma IV.7.

**Lemma V.1 (Distribution of outstanding demands)** *Suppose the generation of demands commences at time 0 and no demands are serviced in the interval  $[0, t]$ . Let  $\mathcal{Q}$  denote the set of all demands in  $[0, W] \times [0, vt]$  at time  $t$ . Then, given a Borel measurable set  $\mathcal{R}$  of area  $A$  contained in  $[0, W] \times [0, vt]$ ,*

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = \frac{e^{-\bar{\lambda}A} (\bar{\lambda}A)^n}{n!}, \quad \text{where } \bar{\lambda} := \lambda/(vW).$$

*Proof:* We first establish the result for a rectangle. Let  $\mathcal{R} = [\ell, \ell + \Delta\ell] \times [h, h + \Delta h]$  be a rectangle contained in  $[0, W] \times [0, vt]$  with area  $A = \Delta\ell\Delta h$ . Let us calculate the probability that at time  $t$ ,  $|\mathcal{R} \cap \mathcal{Q}| = n$  (that is, the probability that  $\mathcal{R}$  contains  $n$  points in  $\mathcal{Q}$ ). We have

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = \sum_{i=n}^{\infty} \mathbb{P} \left[ i \text{ demands arrived in } \left[ \frac{h}{v}, \frac{h + \Delta h}{v} \right] \right] \times \mathbb{P}[n \text{ of } i \text{ are generated in } [\ell, \ell + \Delta\ell]].$$

Since the generation process is temporally Poisson and spatially uniform the above equation can be rewritten as

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = \sum_{i=n}^{\infty} \mathbb{P} [i \text{ demands arrived in } [0, \Delta h/v]] \times \mathbb{P}[n \text{ of } i \text{ are generated in } [0, \Delta\ell]]. \quad (16)$$

Now we compute

$$\mathbb{P} [i \text{ demands arrived in } [0, \Delta h/v]] = \frac{e^{-\lambda\Delta h/v} (\lambda\Delta h/v)^i}{i!},$$

and

$$\mathbb{P}[n \text{ of } i \text{ are in } [0, \Delta\ell]] = \binom{i}{n} \left( \frac{\Delta\ell}{W} \right)^n \left( 1 - \frac{\Delta\ell}{W} \right)^{i-n},$$

so that, substituting these expressions and adopting the shorthands  $L := \Delta\ell/W$  and  $H := \Delta h/v$ , equation (16) becomes

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = e^{-\lambda H} L^n \sum_{i=n}^{\infty} \frac{(\lambda H)^i}{i!} \binom{i}{n} (1-L)^{i-n}. \quad (17)$$

Rewriting  $(\lambda H)^i$  as  $(\lambda H)^n (\lambda H)^{i-n}$ , and using the definition of binomial  $\binom{i}{n} = \frac{i!}{n!(i-n)!}$ , equation (17) reads

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = e^{-\lambda H} \frac{(\lambda L H)^n}{n!} \sum_{j=0}^{\infty} \frac{(\lambda H(1-L))^j}{j!} = e^{-\lambda H + \lambda H(1-L)} \frac{(\lambda L H)^n}{n!} = e^{-\lambda L H} \frac{(\lambda L H)^n}{n!}.$$

Finally, since  $LH = A/(vW)$ , we obtain

$$\mathbb{P}[|\mathcal{R} \cap \mathcal{Q}| = n] = e^{-\bar{\lambda} A} \frac{(\bar{\lambda} A)^n}{n!},$$

where  $\bar{\lambda} := \lambda/(vW)$ . Thus, the result is established for a closed rectangle. But notice that the result also holds for the case when a rectangle is open. Further, the result holds for a countable union of disjoint rectangles due to the demand generation process being uniform along the generator and Poisson in time. Now, a Borel measurable set on  $[0, W] \times [0, vt]$  can be generated by countable unions and complements of rectangles. But both, the unions and the complements of rectangles can be written as a disjoint union of a countable number of rectangles. Hence, the result holds for any Borel measurable set in  $[0, W] \times [0, vt]$ . ■

**Remark V.2 (Uniformly distributed demands)** Lemma V.1 shows us that the number of demands in an unserved region with area  $A$  is Poisson distributed with parameter  $\lambda A/(vW)$ , and conditioned on this number, the demands are distributed uniformly. □

**Lemma V.3 (Travel time bound)** Consider the set  $\mathcal{Q}$  of demands that are uniformly distributed in  $\mathcal{E}$  at time  $t$ . Let  $T_d$  be a random variable giving the minimum amount of time required to travel to a demand in  $\mathcal{Q}$  from a vehicle position selected a priori. Then

$$\mathbb{E}[T_d] \geq \frac{1}{2} \sqrt{\frac{vW}{\lambda}}.$$

*Proof:* Let  $\mathbf{p} = (X, Y)$  denote the vehicle location selected a priori. To obtain a lower bound on the minimum travel time, we consider the best-case scenario, when no demands have been serviced in the time interval  $[0, t]$ . Consider a demand in  $\mathcal{Q}$  with position  $(x, y)$  at time  $t$ . Using Proposition II.2, we can write the travel time  $T$  from  $\mathbf{p}$  to  $\mathbf{q} := (x, y)$  implicitly as

$$T(\mathbf{p}, \mathbf{q})^2 = (X - x)^2 + ((Y - y) - vT(\mathbf{p}, \mathbf{q}))^2. \quad (18)$$

Next, define the set  $S_T$  as the collection of demands that can be reached from  $(X, Y)$  in  $T$  or fewer time units. From equation (18) we see that when  $v < 1$ , the set  $S_T$  is a disk of radius  $T$  centered at  $(X, Y - vT)$ . That is,

$$S_T := \{(x, y) \in \mathcal{E} \mid (X - x)^2 + ((Y - vT) - y)^2 \leq T^2\},$$

where we have omitted the dependence of  $T$  on  $\mathbf{p}$  and  $\mathbf{q}$ . The area of the set  $S_T$ , denoted  $\text{area}(S_T)$ , is upper bounded by  $\pi T^2$ , and the area is equal to  $\pi T^2$  if the  $S_T$  does not intersect a boundary of  $\mathcal{E}$ . Now, by Lemma V.1

the demands in an unserved region are uniformly randomly distributed with density  $\bar{\lambda} = \lambda/(vW)$ . Let us compute the distribution of  $T_d := \min_{\mathbf{q} \in \mathcal{Q}} T(\mathbf{p}, \mathbf{q})$ . For every vehicle position  $\mathbf{p}$  chosen before the generation of demands, the probability that  $T_d > T$  is given by

$$\mathbb{P}[T_d > T] = \mathbb{P}[|S_T \cap \mathcal{Q}| = 0] = e^{-\bar{\lambda} \text{area}(S_T)} \geq e^{-\lambda \pi T^2 / (vW)},$$

where the second equality is by Lemma V.1, and the last inequality comes from the fact that  $\text{area}(S_T) \leq \pi T^2$  (Note that this is equivalent to assuming that the entire plane  $\mathbb{R}^2$  contains demands with density  $\bar{\lambda}$ ). Hence we have

$$\mathbb{E}[T_d] \geq \int_0^{+\infty} \mathbb{P}[T_d > T] dT \geq \int_0^{+\infty} e^{-\lambda \pi T^2 / (vW)} dT = \frac{\sqrt{\pi}}{2\sqrt{\lambda \pi / (vW)}} = \frac{1}{2} \sqrt{\frac{vW}{\lambda}}.$$

■

We can now prove part (i) of Theorem III.1.

*Proof of part (i) of Theorem III.1:* A necessary condition for the stability of any policy is

$$\lambda \mathbb{E}[T] \leq 1,$$

where  $\mathbb{E}[T]$  is the steady-state expected travel time between demands  $i$  and  $i+1$ . For every policy  $\mathbb{E}[T] \geq \mathbb{E}[T_d] \geq \frac{1}{2} \sqrt{\frac{vW}{\lambda}}$ . Thus, a necessary condition for stability is that

$$\lambda \frac{1}{2} \sqrt{\frac{vW}{\lambda}} \leq 1 \quad \iff \quad \lambda \leq \frac{4}{vW}.$$

■

**Remark V.4 (Constant fraction service)** A necessary condition for the existence of a policy which services a fraction  $c \in ]0, 1]$  of the demands is that

$$\lambda \leq \frac{4}{c^2 vW}.$$

Thus, for a fixed  $v \in ]0, 1[$  no policy can service a constant fraction of the demands as  $\lambda \rightarrow +\infty$ . This follows because in order to service a fraction  $c$  we require that  $c\lambda \mathbb{E}[T_d] < 1$ .

In order to service a fraction  $c$  of the demands, we consider a subset of the generator having length  $cW$ , with the arrival rate on that subset being equal to  $c\lambda$ . The use of the TMHP-based policy on this subset and with the arrival rate  $c\lambda$  gives a sufficient condition for stability analogous to Theorem III.2, but with an extra term of  $c^2$  in the denominator.  $\square$

For the proof of part (ii) of Theorem III.1, we first recall from Lemma IV.6 that for stability of the FCFS policy, although  $\lambda$  must go to zero as  $v \rightarrow 1^-$ , it can go very slowly to 0. Specifically,  $\lambda$  goes to zero as

$$\frac{1}{\sqrt{-\ln(1-v)}}.$$

This quantity goes to zero more slowly than any polynomial in  $(1-v)$ . We are now ready to complete the proof of Theorem III.1.

*Proof of part (ii) of Theorem III.1:* The central idea of this proof is that in the limit as  $v \rightarrow 1^-$ , if the vehicle skips a demand and services the next one at an instance, then to reach the demand it skipped, the vehicle has to travel infinitely far from the generator which leads to instability.

Observe that the condition on  $\lambda$  in the statement of part (ii) is the expression given by the necessary condition for FCFS stability in the asymptotic regime as  $v \rightarrow 1^-$ , from Lemma IV.6. Therefore, suppose there is a policy  $P$  that does not serve demands FCFS, but can stabilize the system with

$$\lambda = B(1 - v)^p,$$

for some  $p > 0$ , and  $B > 0$ . Let  $t_i$  be the first instant at which policy  $P$  deviates from FCFS. Then, the demand served immediately after  $i$  is demand  $i + k$  for some  $k > 1$ . When the vehicle reaches demand  $i + k$  at time  $t_{i+1}$ , demand  $i + 1$  has moved above the vehicle. To ensure stability, demand  $i + 1$  must eventually be served. The time to travel to demand  $i + 1$  from any demand  $i + j$ , where  $j > 1$ , is

$$\begin{aligned} T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) &= \sqrt{\left(\frac{\Delta x}{\sqrt{1-v^2}}\right)^2 + \left(\frac{\Delta y}{1-v^2}\right)^2} + \frac{v\Delta y}{1-v^2} \\ &\geq \frac{\Delta y}{1-v^2} + \frac{v\Delta y}{1-v^2} = \frac{\Delta y}{1-v}, \end{aligned}$$

where  $\Delta x$  and  $\Delta y$  are now the minimum of the  $x$  and  $y$  distances from  $\mathbf{q}_{i+j}$  to the  $\mathbf{q}_{i+1}$ . The random variable  $\Delta y/v$  is Erlang distributed with shape  $j - 1 \geq 1$  and rate  $\lambda$ , implying

$$\mathbb{P}[\Delta y \leq c] \leq 1 - e^{-\lambda c/v}, \quad \text{for each } c > 0, \text{ and in particular, for } c = (1 - v)^{1/2-p}.$$

Now, since  $\lambda = B(1 - v)^p$  as  $v \rightarrow 1^-$ ,  $\Delta y > (1 - v)^{1/2-p}$ , with probability one. Thus

$$T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) \geq (1 - v)^{-(p+1/2)},$$

with probability one, as  $v \rightarrow 1^-$ . Thus, the expected number of demands that arrive during  $T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1})$  is

$$\lambda T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) \geq B(1 - v)^p (1 - v)^{-(p+1/2)} \geq B(1 - v)^{-1/2} \rightarrow +\infty,$$

as  $v \rightarrow 1^-$ . This implies that with probability one, the policy  $P$  becomes unstable when it deviates from FCFS and that any deviation must occur with probability zero as  $v \rightarrow 1^-$ . Thus, a necessary condition for a policy to be stabilizing with  $\lambda = B(1 - v)^p$  is that, as  $v \rightarrow 1^-$ , the policy must serve demands in the order in which they arrive. But this needs to hold for every  $p$  and, by letting  $p$  go to infinity,  $B(1 - v)^p$  converges to zero for all  $v \in (0, 1]$ . Thus, a non-FCFS policy cannot stabilize the system no matter how quickly  $\lambda \rightarrow 0^+$  as  $v \rightarrow 1^-$ . Hence, as  $v \rightarrow 1^-$ , every stabilizing policy must serve the demands in the order in which they arrive. Additionally, notice that the definition of the FCFS policy is that it uses the minimum time control (i.e., constant bearing control) to move between demands, thus the expression in part (ii) of Theorem III.1 is a necessary condition for all stabilizing policies as  $v \rightarrow 1^-$ . ■

### B. Proofs of Theorem III.2 and Theorem III.4

We first present the proof of Theorem III.2. We begin with the proof of part (i).

*Proof of part (i) of Theorem III.2:* The central idea is to derive a recurrence relation between the expected  $Y$  coordinate of the vehicle at the next iteration and the expected  $Y$  coordinate at the present iteration by using the background results on the computation of the TMHP in the iteration. The stability condition follows by ensuring that the resulting evolution of the expected  $Y$  coordinate with the number of iterations remains bounded.

Note that if there are any demands “above” the vehicle initially, at the end of the first iteration of the TMHP-based policy, all outstanding demands have their  $y$ -coordinates less than or equal to that of the vehicle, and hence would be located “below” the vehicle as shown in the first of Figure 4. Hence at the end of every iteration of the TMHP-based policy, all outstanding demands would be located “below” the vehicle.

Let the vehicle be located at  $\mathbf{p}(t_i) = (X(t_i), Y(t_i))$  and  $\mathbf{q}_{\text{last}}$  denote the demand with the least  $y$ -coordinate at time instant  $t_i$ . Let  $|\mathcal{Q}|$  denote the number of demands in the set  $\mathcal{Q}$ . If there exists a non-empty set of unserved demands  $\mathcal{Q}$  below the vehicle at time  $t_i$ , then we have

$$Y(t_{i+1}) = v\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1(t_i), \dots, \mathbf{q}_{\text{last}-1}(t_i)\}, \mathbf{q}_{\text{last}}(t_i)) + y_{\text{last}}(t_i), \quad \text{w.p. } \mathbb{P}(|\mathcal{Q}| = k), \text{ for } k \in \{1, 2, \dots\},$$

where  $y_{\text{last}}(t_i)$  is the  $y$ -coordinate of  $\mathbf{q}_{\text{last}}(t_i)$  and  $\mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1(t_i), \dots, \mathbf{q}_{\text{last}-1}(t_i)\}, \mathbf{q}_{\text{last}}(t_i))$  is the time taken for the vehicle as per the TMHP that begins at  $\mathbf{p}(t_i)$ , serves all demands in  $\mathcal{Q}$  other than  $\mathbf{q}_{\text{last}}$  and ends at the demand  $\mathbf{q}_{\text{last}}$ .

We seek an upper bound for the length  $\mathcal{L}_{T,v}$  of the TMHP for which we use the Convert-to-EMHP method (cf. Section II-B). Invoking Lemma II.5 for  $\mathcal{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_{\text{last}-1}\}$ , and writing  $Y_i := Y(t_i)$  for convenience, we have

$$\begin{aligned} \mathcal{L}_{T,v}(\mathbf{p}(t_i), \{\mathbf{q}_1(t_i), \dots, \mathbf{q}_{\text{last}-1}(t_i)\}, \mathbf{q}_{\text{last}}(t_i)) \\ &= \mathcal{L}_E(\text{cnvrt}_v(\mathbf{p}(t_i)), \{\text{cnvrt}_v(\mathbf{q}_1(t_i)), \dots, \text{cnvrt}_v(\mathbf{q}_{\text{last}-1}(t_i))\}, \text{cnvrt}_v(\mathbf{q}_{\text{last}}(t_i))) + \frac{v(y_{\text{last}}(t_i) - Y_i)}{1 - v^2} \\ &\leq \sqrt{\frac{2W(Y_i - y_{\text{last}}(t_i))|\mathcal{Q}|}{(1 - v^2)^{3/2}}} + \frac{Y_i - y_{\text{last}}(t_i)}{1 + v} + \frac{5W}{2\sqrt{1 - v^2}}, \\ &\leq \sqrt{\frac{2WY_i|\mathcal{Q}|}{(1 - v^2)^{3/2}}} + \frac{Y_i}{1 + v} + \frac{5W}{2\sqrt{1 - v^2}}, \end{aligned}$$

where the first inequality is obtained using Lemma II.3, and the second inequality follows since  $y_{\text{last}}(t_i) \geq 0$ . Thus, when  $\mathcal{Q}$  is non-empty at time  $t_i$ ,

$$Y_{i+1} = v\mathcal{L}_{T,v} + y_{\text{last}}(t_i).$$

If  $\mathcal{Q}$  is empty at time  $t_i$ , then the vehicle moves towards the optimal location  $(X^*, Y^*)$ . When a new demand arrives, the vehicle moves towards it. If  $Y_i \leq W$ , then in the worst-case, the vehicle is very close to an endpoint of the generator and the next demand arrives at the other endpoint. In this case, the vehicle moves with a vertical velocity component equal to  $v$  and horizontal component equal to  $\sqrt{1 - v^2}$ . So in the worst-case, the vehicle is at a height  $vW/\sqrt{1 - v^2}$  at the beginning of the next iteration. The other possibility is if  $Y_i > W$ . In this case, to get an upper bound on the height of the vehicle at the next iteration, we consider the vehicle motion when it first



moves horizontally so that the  $x$ -coordinate equals that of the demand, and then moves vertically down to meet the demand. This gives an upper bound on the height of the vehicle at the next iteration as  $v(Y_i - vW)/(1 + v)$ . Thus, if  $\mathcal{Q}$  is empty, then the sum of these two upper bounds is trivially an upper bound on the height of the vehicle at the beginning of the next iteration. Thus, if  $\mathcal{Q}$  is empty, then

$$Y_{i+1} \leq \frac{vW}{\sqrt{1-v^2}} + \frac{v}{1+v}(Y_i - vW) \leq \frac{vW}{\sqrt{1-v^2}} + \frac{vY_i}{1+v}.$$

Conditioned on  $Y_i$ , we have

$$\begin{aligned} \mathbb{E}[Y_{i+1}|Y_i] &\leq \left(\frac{vW}{\sqrt{1-v^2}} + \frac{vY_i}{1+v}\right)\mathbb{P}(|\mathcal{Q}|=0|Y_i) + \\ &\quad v \sum_{k=1}^{\infty} \left(\sqrt{\frac{2WY_i k}{(1-v^2)^{3/2}}} + \frac{Y_i}{1+v} + \frac{5W}{2\sqrt{1-v^2}}\right)\mathbb{P}(|\mathcal{Q}|=k|Y_i) + \mathbb{E}[y_{\text{last}}(t_i)|Y_i]. \end{aligned}$$

It can be shown that  $\mathbb{E}[y_{\text{last}}(t_i)|Y_i] \leq v/\lambda$ . Collecting the terms with  $vY_i/(1+v)$  together and on further simplifying, we obtain

$$\begin{aligned} \mathbb{E}[Y_{i+1}|Y_i] &\leq \frac{vW}{\sqrt{1-v^2}}\mathbb{P}(|\mathcal{Q}|=0|Y_i) + \frac{vY_i}{1+v} + \sum_{k=1}^{\infty} \left(\sqrt{\frac{2v^2WY_i k}{(1-v^2)^{3/2}}} + \frac{5vW}{2\sqrt{1-v^2}}\right)\mathbb{P}(|\mathcal{Q}|=k|Y_i) + \frac{v}{\lambda} \\ &\leq \frac{vW}{\sqrt{1-v^2}} + \frac{vY_i}{1+v} + \sqrt{\frac{2v^2W}{(1-v^2)^{3/2}}}\mathbb{E}[\sqrt{|\mathcal{Q}|Y_i}|Y_i] + \frac{5vW}{2\sqrt{1-v^2}} \sum_{k=1}^{\infty} \mathbb{P}(|\mathcal{Q}|=k|Y_i) + \frac{v}{\lambda} \\ &\leq \frac{vY_i}{1+v} + \sqrt{\frac{2v^2W}{(1-v^2)^{3/2}}}\sqrt{Y_i}\mathbb{E}[\sqrt{|\mathcal{Q}|}|Y_i] + \frac{7vW}{2\sqrt{1-v^2}} + \frac{v}{\lambda}. \end{aligned} \quad (19)$$

Applying Jensen's inequality to the conditional expectation in the second term in the right hand side of equation (19), we have

$$\mathbb{E}[\sqrt{|\mathcal{Q}|}|Y_i] \leq \sqrt{\mathbb{E}[|\mathcal{Q}||Y_i]} = \sqrt{\lambda \frac{Y_i}{v}},$$

where the equality follows since the arrival process is Poisson with rate  $\lambda$  and for a time interval  $Y_i/v$ . Substituting into equation (19), we obtain

$$\mathbb{E}[Y_{i+1}|Y_i] \leq \left(\frac{v}{1+v} + \sqrt{\frac{2v\lambda W}{(1-v^2)^{3/2}}}\right)Y_i + \frac{7vW}{2\sqrt{1-v^2}} + \frac{v}{\lambda}.$$

Using the law of iterated expectation, we have

$$\mathbb{E}[Y_{i+1}] = \mathbb{E}[\mathbb{E}[Y_{i+1}|Y_i]] \leq \left(\frac{v}{1+v} + \sqrt{\frac{2v\lambda W}{(1-v^2)^{3/2}}}\right)\mathbb{E}[Y_i] + \frac{7vW}{2\sqrt{1-v^2}} + \frac{v}{\lambda}, \quad (20)$$

which is a linear recurrence in  $\mathbb{E}[Y_i]$ . Thus,  $\lim_{i \rightarrow +\infty} \mathbb{E}[Y_i]$  is finite if

$$\frac{v}{1+v} + \sqrt{\frac{2Wv\lambda}{(1-v^2)^{3/2}}} < 1 \quad \iff \quad \lambda < \frac{(1-v^2)^{3/2}}{2Wv(1+v)^2}.$$

Thus, if  $\lambda$  satisfies the above condition, then expected number of demands in the environment is finite and the TMHP-based policy is stable.

Finally, from Lemma IV.1, the region of stability for the FCFS policy is contained in the region of stability for the TMHP-based policy. Thus, the TMHP-based policy is stable for all arrival rates the FCFS policy is contained

in the region of stability for the TMHP policy. Thus, the TMHP-based policy is stable for all arrival rates satisfying the bound in Lemma IV.7. This gives us the desired result. ■

**Remark V.5 (Upper bound on expected delay)** Since equation (20) is a linear recurrence in  $\mathbb{E}[Y_i]$ , we can easily obtain an upper bound for  $\lim_{i \rightarrow +\infty} \mathbb{E}[Y_i]$ . Moreover, we may upper bound the expected delay for a demand by

$$\left( \frac{7W}{2\sqrt{1-v^2}} + \frac{v}{\lambda} \right) \left( \frac{1}{1/(1+v) - \sqrt{2Wv\lambda/(1-v^2)^{3/2}}} \right). \quad \bullet$$

*Proof of part (ii) of Theorem III.2:* As the arrival rate  $\lambda \rightarrow +\infty$ , the necessary condition in Theorem III.1 states that for stability,  $v$  must tend to zero. Now, for this part, we make use of the following two facts. First, as  $v \rightarrow 0^+$ , the length of the TMHP constrained to start at the vehicle location and end at the lowest demand, is equal to the length of the EMHP in the corresponding static instance under the map  $\text{cnvrt}_v$ , as described in Lemma II.5. Second, consider a set  $\mathcal{Q}$  of  $n$  points which are uniformly distributed in a region with finite area. Then, in the limit as  $n \rightarrow +\infty$ , the length of a constrained EMHP through  $\mathcal{Q}$  tends to the length of the ETSP tour through  $\mathcal{Q}$ .

More specifically, consider the  $i$ th iteration of the TMHP-based policy, and let  $Y_i > 0$  be the position of the service vehicle. In the limit as  $\lambda \rightarrow +\infty$ , the number of outstanding demands in that iteration  $n_i \rightarrow +\infty$ , and in addition, conditioned on  $n_i$ , the demands are uniformly distributed in the region  $[0, W] \times [0, Y_i]$  (cf. Remark V.2). Now using the above two facts, we can apply Theorem II.4 to obtain an expression for the length of the TMHP constrained to start at the vehicle location and ending at the lowest demand. As  $\lambda \rightarrow +\infty$ , the position of the vehicle at the end of the  $i$ th iteration is given by

$$Y_{i+1} = v\beta_{\text{TSP}}\sqrt{n_i A} = v\beta_{\text{TSP}}\sqrt{n_i Y_i W},$$

with probability one, where  $A := Y_i W$  is the area of the region below the vehicle at the  $i$ th iteration. Thus, conditioned on  $Y_i$  being bounded away from 0, we have

$$\mathbb{E}[Y_{i+1}|Y_i] = v\beta_{\text{TSP}}\sqrt{Y_i}\mathbb{E}[\sqrt{Wn_i}] \leq v\beta_{\text{TSP}}\sqrt{WY_i}\mathbb{E}[n_i],$$

with probability one, where we have applied Jensen's inequality. Using Lemma V.1,  $\mathbb{E}[n_i] = WY_i\lambda/(vW)$ . Thus, with probability one,

$$\mathbb{E}[Y_{i+1}|Y_i] \leq v\beta_{\text{TSP}}\sqrt{W^2 Y_i^2 \frac{\lambda}{vW}} = \beta_{\text{TSP}}\sqrt{\lambda v W Y_i}.$$

Thus, the sufficient condition for stability of the TMHP-based policy as  $\lambda \rightarrow +\infty$  (and thus  $v \rightarrow 0^+$ ) is

$$v < \frac{1}{\beta_{\text{TSP}}^2 W \lambda} \approx \frac{1.9726}{W \lambda}.$$

Finally, we present the proof of Theorem III.4. ■

*Proof of Theorem III.4:* It follows from Lemma IV.3 and Lemma IV.1 that the FCFS minimizes the steady state expected delay in the asymptotic regime of low arrival. Part (i) of Theorem III.4 follows since in the asymptotic regime of low arrival, the TMHP-based policy becomes equivalent to the FCFS policy. The proof of part (ii)

follows from part (ii) of Theorem III.1 and Lemma IV.1 along with the fact that the TMHP-based policy spends the minimum amount of time to travel between demands. ■

## VI. SIMULATIONS

In this section, we present a numerical study of the TMHP-based policy. We numerically determine the region of stability of the TMHP-based policy, and compare it with the theoretical results from the previous sections.

In the actual implementation of the TMHP-based policy, the computational complexity increases undesirably as the values of the problem parameters  $(\lambda, v)$  approach the instability region. Therefore, we adopt a different procedure to characterize the stable/unstable region boundary, which is based upon the following central idea. For a given value of  $(\lambda, v)$  and a sufficiently high value of the height of the vehicle, if the policy is stable, then after one iteration of the policy, the vehicle's height must decrease. In particular, the following procedure was adopted.

- (i) For a collection of instructive pairs of the demand speed  $v$  and  $\lambda$  in the region of interest, we set the generator width  $W = 1$  and we set the initial height  $h_0$  of the environment of interest so that the expected number of demands in the environment are 1000. Thus,  $h_0 = 1000v/\lambda$ .
- (ii) We repeated 10 times the following procedure. The vehicle is placed at the height  $h_0$  and at a uniformly random location in the horizontal direction. A Poisson distributed number  $n_0$  with parameter  $\lambda/v$ , of outstanding demands are uniformly randomly placed in the environment (cf. Lemma V.1). The vehicle uses the TMHP-based policy to serve all outstanding demands and we store the height  $h_1$  of the vehicle at the end of the single iteration of the policy. Finally, we compute the average height  $\bar{h}_1$  of the 10 iterations.
- (iii) If  $\bar{h}_1 \leq h_0$ , then the policy is deemed to be stable for the chosen value of  $(v, \lambda)$ . Otherwise the policy is deemed to be unstable.

The `linkern`<sup>1</sup> solver was used to generate approximations to the TMHP at every iteration of the policy. The `linkern` solver takes as an input an instance of the Euclidean travelling salesperson problem. To transform the constrained EMHP problem into an ETSP, we replace the distance between the start and end points with a large negative number, ensuring that this edge is included in the `linkern` output.

The results of this numerical experiment are presented in Figure 8. For the purpose of comparison, we overlay the plots for the theoretical curves, which were presented in Figure 1. We observe that the numerically obtained stability boundary for the TMHP-based policy falls between the necessary and the sufficient conditions which were established in parts (i) of Theorems III.1 and III.2 respectively. Further, although the sufficient condition characterized in part (ii) of Theorem III.2, is theoretically an approximation of the stability boundary in the asymptotic regime of high arrival, our numerical results show that the condition serves as a very good approximation to the stability boundary, for nearly the entire range of demand speeds.

For different values of  $(\lambda, v)$  in the stable region, we present simulations of the TMHP-based policy that shed light on the steady state height of the vehicle. For three different values of  $v$ , we empirically determine the steady

<sup>1</sup>The TSP solver `linkern` is freely available for academic research use at <http://www.tsp.gatech.edu/concorde.html>.

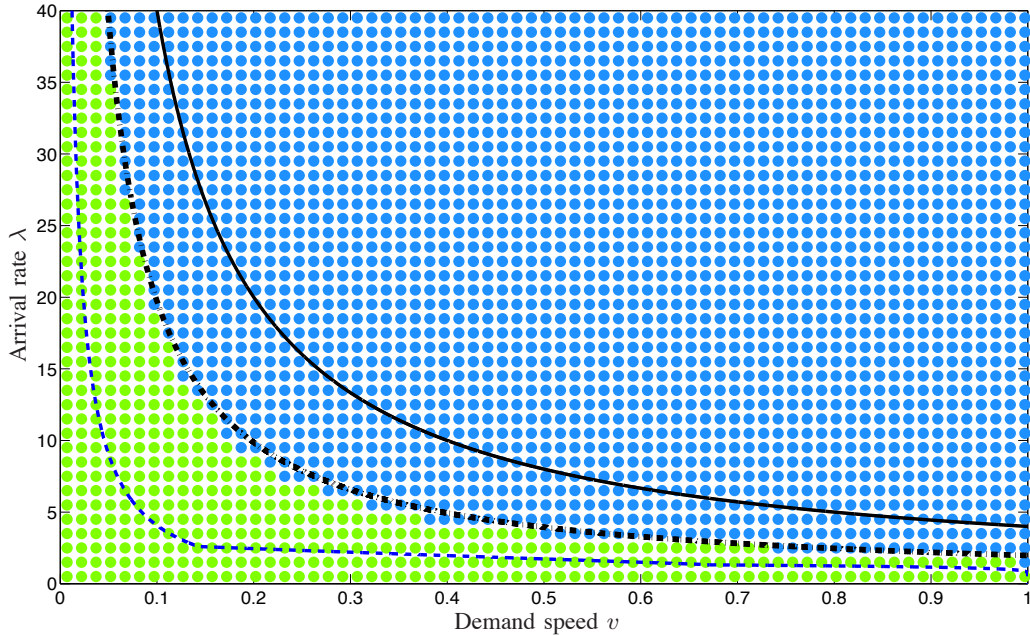


Fig. 8. Numerically determined region of stability for the TMHP-based policy. A lightly shaded (green-coloured) dot represents stability while a darkly shaded (blue-coloured) dot represents instability. The uppermost (thick solid) curve is the necessary condition for stability for any policy as derived in Theorem III.1. The lowest (dashed) curve is the sufficient condition for stability of the TMHP-based policy as established by Theorem III.2. The broken curve between the two curves is the sufficient stability condition of the TMHP-based policy in the high arrival regime as derived in part (ii) of Theorem III.2. The environment width is  $W = 1$ .

state vehicle height at various arrival rates. The results are shown in Figure 9. We observe that the steady state height increases (i) at higher arrival rates for a fixed demand speed, and (ii) at higher demand speeds at fixed arrival rates, which is fairly intuitive to expect for a stable policy.

## VII. CONCLUSION AND FUTURE DIRECTIONS

We introduced a dynamic vehicle routing problem with translating demands. We determined a necessary condition on the arrival rate of the demands for the existence of a stabilizing policy. In the limit when the demands move as fast as the vehicle, we showed that every stabilizing policy must service the demands in the FCFS order. We proposed a novel receding horizon policy that services the moving demands as per a translational minimum Hamiltonian path. In the asymptotic regime when the demands move as fast as the vehicle, we showed that the TMHP-based policy minimizes the expected time to service a demand. We derived a sufficient condition for stability of the TMHP-based policy, and showed that in the asymptotic regime of low demand speed, the sufficient condition is within a constant factor of the necessary condition for stability. In a third asymptotic regime when arrival rate tends to zero for a fixed demand speed, we showed that the TMHP-based policy is optimal in terms of minimizing the expected time to service a demand. Finally, we presented an implementation of the TMHP-based policy to

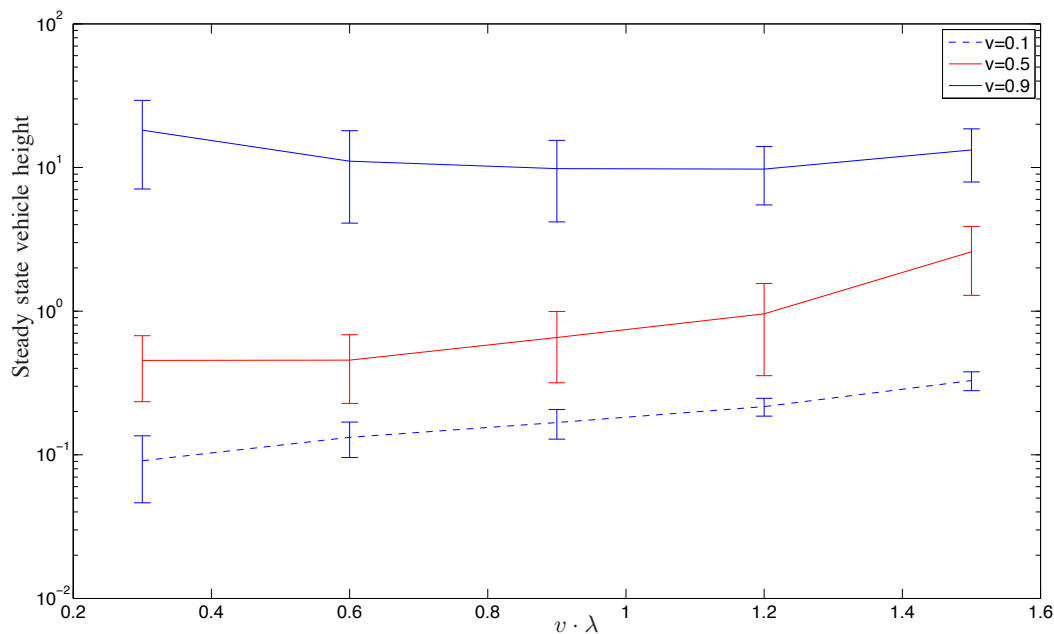


Fig. 9. Variation of the steady state vehicle height with the product  $v \cdot \lambda$  at different demand speeds. The error bars correspond to  $\pm 1$  standard deviation about the mean.

numerically determine its region of stability. Our numerical results show that the theoretically established sufficient condition in the asymptotic regime of low demand speeds also serves as a good approximation to the boundary of the stability region for a significantly large interval of values of demand speed.

As future work, it would be of interest to extend the present analysis or to design an improved policy that is optimal also in the asymptotic regime of high arrival, in addition to the two other asymptotic regimes. Another interesting direction is to have on-site service times for the demands. For the case in which the on-site service times are independent and identically distributed with a known expected value, and the vehicle is permitted to move with the demand upon reaching it, the results in this paper could be extended using similar analysis. More recently, in [34], we have addressed a version of the present problem in which the goal for the vehicle is to maximize the fraction of demands served before they reach a deadline, which is at a given distance from the generator. We also plan to address versions of the present problem that involve a non-uniform spatial arrival density and multiple service vehicles.

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## APPENDIX

In this Appendix, we present the proof of Lemma II.3.

*Proof:* Suppose the rectangular region is given by  $0 \leq x \leq 1$ ,  $0 \leq y \leq h$ . Let  $m$  be a positive integer (to be chosen later) and let the  $n$  points be denoted by  $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ . We now construct two paths through the points. The first consists of (a) the  $m + 1$  lines  $y = 0, h/m, 2h/m, \dots, h$ ; (b) the  $n$  shortest distances from each of the  $n$  points to the nearest such line, each traveled twice, and (c) suitable portions of the lines  $x = 0$ ,  $0 \leq y \leq h$ , and  $x = 1$ ,  $0 \leq y \leq h$ . This is illustrated in Figure 10. The length of this path is

$$l_1 = m + 1 + 2 \sum_{i=1}^n d_1(\mathbf{q}_i) + h,$$

where the notation  $d_1(\mathbf{q}_i)$  denotes the shortest distance of point  $\mathbf{q}_i$  from the nearest of the  $m + 1$  lines. The second path is constructed similarly using the  $m$  lines  $y = h/2m, 3h/2m, \dots, (2m - 1)h/2m$ . This path also commences on  $y = h$ , passes through the above  $m$  lines (visiting the points whenever they are at the shortest distance from these  $m$  lines) and ends on  $y = 0$ . The length of this path is

$$l_2 = (m + 2) + 2 \sum_{i=1}^n d_2(\mathbf{q}_i) + h,$$

where the notation  $d_2(\mathbf{q}_i)$  denotes the shortest distance of point  $\mathbf{q}_i$  from the nearest of the new  $m$  lines.

Observe that  $d_1(\mathbf{q}_i) + d_2(\mathbf{q}_i) = h/2m$ . Hence,

$$l_1 + l_2 = 2m + 3 + 2h + hn/m.$$

Now choose  $m$  to be the integer nearest to  $\sqrt{hn}/2$ , so that  $n = 2(m + \theta)^2/h$ , where  $|\theta| \leq 1$ . Thus,

$$\begin{aligned} l_1 + l_2 &= 2m + 3 + 2h + 2(m + \theta)^2/m \\ &= 4(m + \theta) + 2h + 3 + 2\theta^2/m \\ &\leq 2\sqrt{2hn} + 2h + 5. \end{aligned}$$

Thus, at least one of the two paths must have length upper bounded by  $\sqrt{2hn} + h + 5/2$ . ■

