A Complete Algorithm for the Infinite Horizon Sensor Scheduling Problem

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***Abstract*— In this paper we study the problem of scheduling sensors to estimate the state of a linear dynamical system. The estimator is a Kalman filter and we seek to optimize the *a posteriori* error covariance over an infinite time horizon. We characterize the exact conditions for the existence of a schedule with uniformly bounded estimation error covariance. Using this result, we construct a scheduling algorithm that guarantees that the error covariance will be bounded if the existence conditions are satisfied. We call such an algorithm complete. We also show that the error will die out exponentially for any detectable LTI system. Finally, we provide simulations to compare the performance of the algorithm against other known techniques.**

I. Introduction

One technique for monitoring an environmental process is to deploy a sensor network. Sensor networks have been used in various applications including determining a robot's state [1], tracking the position of a target [2], selecting the frequency in radar and sonar applications, or monitoring tasks such as chemical processes [3], seismic activity or toxin levels at a factory. Sensor scheduling techniques can also be applied to problems such as adaptive compressed sensing [4].

The collection of data can be done by operating every sensor continuously; however, when the network has strict energy constraints this strategy may not be viable. To overcome these restrictions, sensors can alternate between awake and asleep modes. To avoid having an incomplete picture of the phenomenon of interest, the sensing schedule should be optimized to maximize the information obtained. This is, in essence, the sensor scheduling problem.

The sensor scheduling problem has received considerable attention in recent years. In the context of linear Gaussian systems, a Kalman filter is the optimal estimator in that it produces an estimate with the least mean square error. Thus, the Kalman filter is commonly used as the basis for the sensor scheduling problem. With this setting, the infinite horizon sensor scheduling problem is studied in [5]. Under some mild conditions, it is shown that the optimal infinite horizon schedule is independent of the initial covariance. It is also shown that the optimal cost can be estimated arbitrarily closely by a periodic schedule, with a finite period, and that such a schedule implies that the error covariance approaches a unique limit cycle. An optimal and semi-optimal algorithm that use tree pruning techniques are provided in [6].

Numerous approaches have been proposed to tackle the sensor scheduling problem. In [7], [8], three different approaches (sliding window, greedy thresholding and random selection) are empirically compared; the random selection method is further analyzed and bounded. In [9] approaches such as a best step look ahead algorithm, an approach based on the Viterbi algorithm and by casting the problem as a duality problem, are studied.

Several methods rely on convex relaxations [10], [11]. However, in [12], [13] the authors show that a greedy algorithm empirically performs better than the method presented in [11]. The authors also show that the greedy algorithm gives good performance guarantees on the maximum *a posteriori* covariance over a single time step of the sensor schedule. A general framework that frames the sensor scheduling problem as a relaxed quadratic program is presented in [14]. A number of problems that incorporate various network constraints can be addressed using this framework. The framework is used to describe and analyze a greedy approach though the error bound presented is not necessarily uniformly bounded for unstable systems.

Most of the approaches that exist in literature consider the optimization problem but either give only empirical results or bounds that are themselves unbounded over time. We do not know of any results that attempt to characterize when any particular sensor schedule will result in a uniformly bounded sequence of covariance matrices.

Contributions: We give necessary and sufficient conditions for the existence of an infinite horizon sensor schedule with a bounded error covariance, which makes a novel connection to detectability. We then provide a complete algorithm for sensor scheduling: That is, our algorithm outputs a uniformly bounded sensor schedule if one exists. The algorithm is a simple modification to a greedy algorithm.

Organization: We give some background in Section II and we then formally define our problem in Section III. We give necessary and sufficient conditions for the existence of a uniformly bounded schedule in Section IV and then show how a greedy algorithm can be modified to take advantage of these conditions in Section V. In SectionVI, we empirically evaluate the proposed algorithm.

II. Preliminaries

A. Review of Linear Algebra Concepts

Given a square matrix $A \in \mathbb{R}^{n\times n}$, we will refer to the **null space** of A as $\mathcal{N}(A)$. The identity matrix will be referred to as I. We will use the notation $\mathrm{col}\,(M_1, \dots, M_k)$ to refer to the matrix formed by stacking the matrices M_1, …, M_k (all of which have the same number of columns).

Finally, we will use the following lemma relating to linearly independent (LI) vectors for some of our proofs.

Lemma II.1. *Given a full column rank matrix $A \in \mathbb{R}^{m\times n}$ and $k \le n$ LI vectors $\{x_i\}_{i=1}^k$. Then $\{Ax_i\}_{i=1}^k$ are also LI.*

This research is partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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B. Observability and Detectability

Consider the discrete-time linear time varying (LTV) system

$$x_{t+1} = A_t x_t, \quad y_t = C_t x_t.$$

where $x_t \in \mathbb{R}^n$ and $y_t \in \mathbb{R}^m$ We first look at the linear time invariant (LTI) case, i.e., $A_t = A$ and $C_t = C$. The system is **observable** if the value of the initial state can be determined given a sequence of measurements, y_k.

Lemma II.2 (Observability)**.** *An LTI system (A, C) is observable if and only if its **observability matrix**, Θ, has rank n, where $\Theta := \operatorname{col}\left(C, CA, \ldots, CA^{n-1}\right)$.*

If the observability matrix is not full rank, i.e., $\mathcal{N}(\Theta) \neq \emptyset$, then the system can be decomposed into observable and unobservable components. This is known as the **standard form for unobservable systems**. Let $T = \begin{bmatrix} T_{\bar{o}} & T' \end{bmatrix}$ such that $T_{\bar{o}}$ is a matrix whose columns form a basis for $\mathcal{N}(\Theta)$ and T' is such that $\operatorname{rank}(T) = n$.

$$\begin{aligned} z = T^{-1}x = \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} &\implies z_{k+1} = \bar{A} z_k, \quad y_k = \bar{C} z_k \\ \bar{A} = T^{-1}AT = \begin{bmatrix} A_{\bar{o}} & A_{12} \\ 0 & A_o \end{bmatrix}&, \quad \bar{C} = CT = \begin{bmatrix} 0 & C^o \end{bmatrix}, \end{aligned} \tag{1}$$

Since (1) is just a similarity transform, the system is equivalent to the original system and the eigenvalues of A and $\bar{A}$ coincide. The advantage of this transform is that (A_o, C^o) is observable whereas $(A_{\bar{o}}, 0)$ is not. Also, if Θ, $\bar{\Theta}$ and Θ^o are the observability matrices of (A, C), $(\bar{A}, \bar{C})$ and (A_o, C^o) respectively, then $\bar{\Theta} = \Theta T = \begin{bmatrix} 0 & \Theta^o \end{bmatrix}$.

Since the stable modes of a system die out exponentially, for many practical purpose it suffices for only the unstable modes to be observable. This generalization of observability is known as detectability.

Definition II.3 (Detectability)**.** The following are equivalent for a LTI system: 1) (A, C) is detectable. 2) $A_{\bar{o}}$ is stable, i.e., no unstable mode is unobservable. 3) For every eigenvector v of A associated with an unstable eigenvalue, $Cv \neq 0$. ■

Coming back to the general case of a LTV system, the State Transition Matrix (STM) for $t_2 \geq t_1$ is $\Phi_{t_2,t_1} = \Phi_{t_2,t_2-1}\Phi_{t_2-1,t_1}$ where $\Phi_{t+1,t} = A_t$. We can define the **sequence observability matrix**,

$$B(t, t+k) = \operatorname{col}\left(C_t, C_{t+1}\Phi_{t+1,t}, \ldots, C_{t+k}\Phi_{t+k,t}\right),$$

as well as the observability **Gramian**,

$$\begin{aligned} X(t, t+k) &= \sum_{i=0}^{k} \Phi_{t+i,t}^\top C_{t+i}^\top C_{t+i} \Phi_{t+i,t} \\ &= B(t, t+k)^\top B(t, t+k). \end{aligned}$$

Although a generalization of observability can be made, we are interested in a slightly stricter notion.

Definition II.4 (Uniform Detectability and Observability)**.** Given a LTV system with STM Φ_{t,t_0} and measurement matrices C_t. The system is uniformly detectable if there exists non-negative integers s, r and constants $\alpha \in [0, 1)$ and $\beta > 0$, such that for all $x \in \mathbb{R}^n$ and all times t,

$$||\Phi_{t+r,t}x|| \geq \alpha ||x|| \implies x^\top X(t, t+s)x \geq \beta ||x||^2. \tag{2}$$

Additionally, the system is uniformly observable if there exists integer s and positive constants β_1, β_2 such that

$$0 \prec \beta_1 I \preceq X(t, t+s) \preceq \beta_2 I,$$

holds in the positive semidefinite sense.[1] ■

Remark II.5 (Stabilizability). Uniform stabilizability can be similarly defined. The definition is omitted here and can be found in [15]. •

III. Problem Formulation and Approach

Consider a linear dynamical system with the state transition matrix given by A and measurement matrix C. Each row of C corresponds to a single sensor in the sensor network. For the sensor scheduling problem, we want to pick a set of k sensors at every time step to make a measurement (i.e., k rows of C). The problem can be represented by the following stochastic LTV system,

$$\begin{aligned} x_{t+1} &= Ax_t + w_t \\ y_{t+1} &= C_t x_t + v_t, \end{aligned} \tag{3}$$

where $t \in \mathbb{Z}_{\geq 0}$, $x_t \in \mathbb{R}^n$ and $y_t \in \mathbb{R}^k$. $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$. The matrix C_t is a subset of k rows of C. This is the standard sensor selection model, as in [6], [14]. For ease of presentation, we focus on the case where $k = 1$, though our results easily extend to the general case. Finally, w_t (process noise) and v_t (measurement noise) are zero mean Gaussian noise vectors with covariance matrices $W, V \in \mathbb{R}^{n \times n}$, respectively, with $W \succeq 0$ and $V \succ 0$. We assume that the noises are independent over time.

A Kalman filter uses noisy measurements to estimate the state in a linear dynamical system; an in depth study can be found in [16]. The Kalman filter gives the best mean squared error of the state estimate among all linear estimators. The following lemma, derived from the results in [15] gives conditions under which the filter is stable, i.e., the expected error of the state estimate goes to zero.

Lemma III.1. *Assume that the system $(A, W^{\frac{1}{2}})$ is uniformly stabilizable. Then the Kalman filter error covariance, $\Sigma_{t|t}$, and predictor covariance, $\Sigma_{t+1|t}$, are bounded if and only if (A, C_t) is uniformly detectable. Furthermore, the Kalman filter is exponentially stable only if (A, C_t) is uniformly detectable.*

For $k = 1$, we choose one sensor at each time step. We can represent a sensor schedule as $\sigma = (\sigma_0, \sigma_1, \ldots)$, where $\sigma_t \in \{1, \ldots, m\}$ is the index of the sensor chosen at time step t. The problem that we consider is the following: under what conditions on A and C does there exist a sensor schedule $\sigma = (\sigma_0, \sigma_1, \ldots)$ that results in the error covariance being bounded? Moreover, how do we construct such a schedule?

[1] A symmetric matrix A is denoted to be positive definite (p.d.) as $A \succ 0$ and positive semi-definite (p.s.d.) as $A \succeq 0$.

It is important that the error covariance be bounded since otherwise the state estimate will never be accurate. Formally, we seek to keep $F(\sigma)$ bounded, for some metric F that measures the covariance.

A. A Modified Detectability Condition

The system under consideration (3) is time varying only due to the measurement sequence (i.e., the sequence of C_t). Thus, we can simplify some of the definitions given in Section II-B. Consider an LTI system (A, C) and a sequence of measurements $\sigma = (\sigma_0, \sigma_1, \ldots)$, and the corresponding sequence of measurement matrices $(C_0, C_1, \ldots)$. For a given time t and time window k, the sequence observability matrix is $B_\sigma(t, t+k) = \operatorname{col}\left(C_t, C_{t+1}A, \ldots, C_{t+k}A^k\right)$. The sequence of measurements σ is uniformly detectable if there exists non-negative integers s, r and constants $\alpha \in [0, 1)$ and $\beta > 0$, such that for all $\{x \in \mathbb{R}^n | \, \|x\| = 1\}$ (without loss of generality since for $x = 0$ the condition is trivially satisfied) and all times t,

$$\|A^r x\| \geq \alpha \implies \|B(t, t+s)x\| \geq \beta > 0. \tag{4}$$

Additionally, the sequence is uniformly observable if there exists integer s and positive constants β_1, β_2 such that

$$\begin{aligned} &0 < \beta_1 \leq \|B(t, t+s)x\| \leq \beta_2 \\ \iff &\operatorname{rank}(B(t, t+s)) = n. \end{aligned} \tag{5}$$

Naturally, a uniformly observable sequence is also uniformly detectable. We now look at when a schedule is uniformly detectable and hence yields a bounded error covariance estimate.

IV. Existence of Uniformly Detectable Sequence

The question we now ask is, given an LTI system, does there exist a sequence of measurements that is uniformly detectable? It is reasonable to expect that if (A, C) is observable, then a sequence of measurements exists such that the system is uniformly observable through that sequence. This, however, is not the case. Consider the trivial example where $A = 0$ (and $\operatorname{rank}(C) = n$). Here, $\operatorname{rank}(B(t, t+s)) = 1$ since the second row onwards will be 0. Note that although the initial state cannot be predicted, the actual progression after a certain time period can, in fact, be determined.

In this section we show that if (A, C) is detectable, then the periodic sequence that sequentially repeats each measurement n times, i.e., each period is $\sigma_C = (1, \ldots, 1, 2, \ldots, 2, \ldots, m, \ldots, m)$ is uniformly detectable.

Theorem IV.1. *If (A, C) is observable and A is full rank, then the system is uniformly observable through the periodic sequence σ_C.*

Proof. Let c_i be the i^{th} row of C. Define,

$$\begin{aligned} \Theta &= \operatorname{col}\left(C, CA, CA^2, \ldots, CA^{n-1}\right) \\ &= \begin{bmatrix} \operatorname{col}(c_1, \ldots, c_m) \\ \operatorname{col}(c_1 A, \ldots, c_m A) \\ \vdots \\ \operatorname{col}\left(c_1 A^{n-1}, \ldots, c_m A^{n-1}\right) \end{bmatrix}, \end{aligned}$$

$$B = \begin{bmatrix} \operatorname{col}\left(c_1, \ldots, c_1 A^{n-1}\right) \\ \operatorname{col}\left(c_2 A^n, \ldots, c_2 A^{2n-1}\right) \\ \vdots \\ c_m A^{mn-1} \end{bmatrix}. \tag{6}$$

In order for the sequence to be uniformly observable, we require the existence of (s, β) such that $\operatorname{rank}(B(t, t{+}s)) = n$ for all t. Take $s = 2mn$ so that the full sequence σ_C shows up in the construction of $B(t, t+s)$; specifically, $B(t, t+s)$ will always contain the rows of BA^p for some p. Since A is full rank, showing $\operatorname{rank}(B) = n$ is sufficient for uniform observability. Since it is given that $\operatorname{rank}(\Theta) = n$, it suffices to show that each of the rows of Θ can be written as a linear combination of the rows in B.

First, define sets to represent the rows of Θ and B. Let $X_i = \{c_i, c_i A, \ldots, c_i A^{n-1}\}$ for $i = 1, \ldots, m$. Note that the rows of Θ comprise of the vectors in the multiset $\bigcup_{i=1}^m X_i$. Let $X_i^b = X_i A^{(i-1)n} = \{c_i A^{(i-1)n}, c_i A^{(i-1)n+1}, \ldots, c_i A^{in-1}\}$. So the rows of B comprise of elements of the multiset $\bigcup_{i=1}^m X_i^b$.

For any particular $1 \leq i \leq m$, there are $k_i \leq n$ linearly independent vectors in X_i. Since A is full rank, the set X_i^b contains k LI vectors too (by Lemma II.1). Also, due to the Cayley-Hamilton theorem, $x \in X_i^b \implies x \in \operatorname{span}(X_i)$. Any k LI vectors in the span of X^i will themselves span the space. As a result, every vector in X_i is in $\operatorname{span}(X_i^b)$. $\square$

Using this result, we now give a constructive proof that shows the existence of a uniformly detectable sequence.

Theorem IV.2. *If (A, C) is detectable, then the periodic sequence of measurements σ_C is uniformly detectable.*

Proof. Let T be defined as in (1). Transforming the system into observable standard form gives the STM $\bar{A}$ with separate observable and unobservable components. Let z and l be the number of zero and stable eigenvalues respectively of the observable component A_o, which is a $d \times d$ matrix. Assume that the generalized eigenvectors of A_o are ordered such that the ones corresponding to the zero eigenvalues come first, then the stable and unstable ones. Let $V := \begin{bmatrix} v_1 & \ldots & v_d \end{bmatrix}$. Consider the following transform.

$$\begin{aligned} u = Q^{-1}x &= \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ where } Q := T \begin{bmatrix} I & 0 \\ 0 & V_{d \times d} \end{bmatrix}_{n \times n} \\ \tilde{A} = Q^{-1}AQ &= \begin{bmatrix} A_{\bar{o}} & A_{12}V \\ 0 & \begin{bmatrix} A_{on} & 0 & 0 \\ 0 & A_{os} & 0 \\ 0 & 0 & A_{ou} \end{bmatrix} \end{bmatrix} \\ \tilde{C} = CQ &= \begin{bmatrix} 0 & C_1 & C_2 & C_3 \end{bmatrix} \end{aligned} \tag{7}$$

Here, A_{os} and A_{ou} have stable and unstable eigenvalues respectively, are both full rank and are composed of Jordan blocks. A_{on} is also composed of Jordan blocks and is nilpotent, so $A_{on}^z = 0$. Also, by definition of detectability, $A_{\bar{o}}$ is stable. For this system, we can define $\tilde{\Theta} = \begin{bmatrix} 0 & \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix}$ and $\tilde{B}(t, t+s) = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \end{bmatrix}$.

Note that $\begin{bmatrix}\Theta_1 & \Theta_2 & \Theta_3\end{bmatrix}$ is full rank since this part corresponds to the observable subsystem. For detectability of the sequence to hold, there should exist (s, r, α, β) such that (4) is satisfied. Following from (4), we consider only initial states of unit norm ($\|u\| = 1$).

Case 1: $u_3 = 0$ or A_{ou} does not exist (i.e., A is stable): In this case, only the stable modes are active and so the state approaches 0 exponentially. As a result, for any $\alpha \in (0, 1)$, there exists $r > 0$ such that $\left\|\tilde{A}^r u\right\| < \alpha$ for all u.

Case 2: $\|u_3\| > 0$: In this case, (s, β) can be chosen so that $\left\|\tilde{B}(t, t+s)u\right\| \geq \beta$ irrespective of the values of (r, α).

Take $s = 2mn$ so that $\tilde{B}(t, t+s)$ always contains the full sequence σ_C. As a result, the *sequence observability matrix* for s time steps, $\tilde{B}(t, t+s)$, will always contain the rows of $\tilde{B}_\sigma \tilde{A}^k$ for some $k > 0$, where $\tilde{B}_\sigma$, defined similarly to (6), can be represented as

$$
\begin{aligned}
\tilde{B}_\sigma &= \operatorname{col}\left(\tilde{c}^1, \dots, \tilde{c}^1 \tilde{A}^{n-1}, \dots, \tilde{c}^m \tilde{A}^{mn-1}\right) \\
&= \begin{bmatrix}
0 & c_1^1 & c_2^1 & c_3^1 \\
\vdots & \vdots & \vdots & \vdots \\
0 & c_1^1 A_{on}^{z-1} & \vdots & \vdots \\
0 & 0 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & c_2^1 A_{os}^{n-1} & c_3^1 A_{ou}^{n-1} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & c_2^m A_{os}^{mn-1} & c_3^m A_{ou}^{mn-1}
\end{bmatrix} \\
&=: \begin{bmatrix}0 & B_1 & B_2 & B_3\end{bmatrix},
\end{aligned}
$$

where $\tilde{c}^i$ is the i^{th} row of $\tilde{C}$. Note that

$$
\tilde{B}_\sigma \tilde{A}^k = \begin{bmatrix}0 & B_1 A_{on}^k & B_2 A_{os}^k & B_3 A_{ou}^k\end{bmatrix}.
$$

Also, $\begin{bmatrix}\Theta_2 & \Theta_3\end{bmatrix}$ is full rank and, since both A_{os} and A_{ou} are full rank, $\begin{bmatrix}B_2 & B_3\end{bmatrix}$ is also full rank (using the same argument as in the proof of Theorem IV.1) and so is $\begin{bmatrix}B_2 A_{os}^k & B_3 A_{ou}^k\end{bmatrix}$. Note that $B_1 A_{on}^k = 0$ for $k \geq z$.

Now, without loss of generality, assume that the sequence σ_C starts at time t. So, since $k = mn \geq z$,

$$
\tilde{B}(t, t+s)u = \begin{bmatrix} B_1 u_1 + B_2 u_2 + B_3 u_3 \\ B_2 A_{os}^{mn} u_2 + B_3 A_{ou}^{mn} u_3 \end{bmatrix} =: \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.
$$

Now, $d_2 = 0$ if and only if $\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0$. Given that $u_3 \neq 0$, it follows that $\left\|\tilde{B}(t, t+s)u\right\| \geq \|d_2\| > 0$. Therefore,

$$
\beta \leq \min_{\{u \mid u_1 = 0, \|u\| = 1\}} \left\| B_2 A_{os}^{mn} u_2 + B_3 A_{ou}^{mn} u_3 \right\|,
$$

is an appropriate choice to obtain detectability. □

Corollary IV.3 (Necessary condition)**.** *If (A, C) is not detectable, then there does not exist a sequence that is uniformly detectable.*

Proof. If (A, C) is not detectable, then there exists an eigenvalue-eigenvector pair, (λ, v), of A such that $|\lambda| \geq 1$ and $Cv = 0$. For any pair (α, r), assuming $\|v\| = 1$, we must have $\|A^r v\| = |\lambda|^r \geq 1 > \alpha$. So, there has to exist (s, β) such that $\|B(t, t+s)v\| \geq \beta$ for uniform detectability. However, the rows of $B(t, t+s)$ consist of vectors of the form $c_i A^k$, for some k, where c_i is a row of C. Now $c_i A^k v = \lambda^k c_i v = 0$ and so $B(t, t+s)v = 0$ no matter what the actual sequence is. Therefore, no sequence of measurements can be uniformly detectable. □

Corollary IV.4 (Multiple sensors)**.** *For the problem of selecting k measurements per time step, a uniformly detectable schedule exists if and only if (A, C) is detectable.*

This result shows that an LTI system is detectable if and only if there exists a sensor schedule constructed from the rows of C that is uniformly detectable (i.e., that yields a uniformly bounded error covariance). We can now formally define the notion of a complete sensor scheduling algorithm.

Definition IV.5 (Complete Sensor Scheduling Algorithm)**.** A sensor scheduling algorithm is complete if for every detectable LTI system (A, C), the resulting sequence of error covariance matrices are uniformly bounded for all time. ■

Although a schedule can be naively constructed using, for example, the periodic sequence σ_C, we now seek an algorithm that attempts to optimize the error covariance while at the same time keeping it uniformly bounded.

V. A Complete Sensor Selection Algorithm

An interesting question to ask is whether or not the greedily constructed schedule is uniformly detectable. The definition of uniform detectability effectively requires that all unstable modes be observable within a certain amount of time. Intuitively, the error associated with an unstable mode will keep growing until some point in time when a measurement corresponding to it should have the most benefit to be chosen. Therefore, a reasonable expectation is that the greedy schedule will lead to a bounded error covariance.

Example V.1. Consider the pathological system

$$
A = \Sigma_0 = I_{3\times 3}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} =: \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},
$$

$$
W = \begin{bmatrix} 0.1 & 0.13 & 0.13 \\ 0.13 & 0.41 & 0.36 \\ 0.13 & 0.36 & 0.33 \end{bmatrix}, \quad V = I_{3\times 3}.
$$

Running the greedy algorithm, the resulting value of the objective function is plotted in Figure 1. The schedule that the greedy algorithm outputs does not select measurement 3 until $t = 8576$. After that, the third sensor is chosen approximately every 73 time steps. Although the schedule is uniformly detectable, the time window needed is over 8000 time steps! ▲

Unfortunately, establishing the boundedness of greedy has proved difficult. The main reason appears to be relating the Kalman update performance metric to the sequence observability matrix. As evident from the previous example,

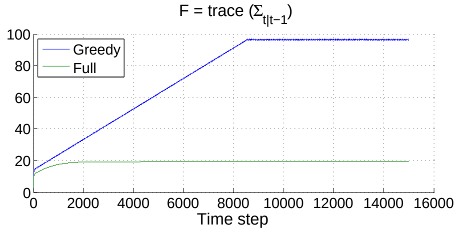


Fig. 1. The value of the greedy sequence over time (cf. Example V.1). The "full" sequence is the value if every measurement is made at all times.

even if the greedy algorithm does produce a uniformly detectable sequence, it may perform quite poorly. We now present a modified greedy algorithm that ensures that the output sequence is uniformly detectable and which attempts to maintain a relatively tight limit for the number of steps required to achieve uniform detectability.

The DETECTABLEGREEDY algorithm is given in Algorithm 1. The idea is to make a greedy choice at each iteration subject to the constraint that the choice of measurement will increase the rank of the matrix M. This matrix emulates the sequence observability matrix $B(t, t+s)$. Once M becomes full rank, it is reset. As a result, M acts as a sliding window and the algorithm attempts to keep this window fully observable. We now show that the algorithm does, in fact, result in a uniformly detectable schedule.

Remark V.2 (Multiple sensors). The same idea can be used for selecting multiple sensors per time step, i.e., giving precedence to measurements that increase the rank of M. The proof of completeness is similar to the one we now give for selecting a single sensor per time step. •

Notation: Parenthesis will be used to refer to a certain time range of M, e.g. $M(2,4)$ refers to measurements made at times 2 though to 4. We also reuse notation from the proof of Theorem IV.2: $A_{su} := \begin{bmatrix} A_{os} & 0 \\ 0 & A_{ou} \end{bmatrix}$, $C_{su} := \begin{bmatrix} C_2 & C_3 \end{bmatrix}$, $\Theta_{su} := \begin{bmatrix} \Theta_2 & \Theta_3 \end{bmatrix}$, $B_{su} := \begin{bmatrix} B_2 & B_3 \end{bmatrix}$ and $p = \operatorname{rank}(A_{su})$.

Lemma V.3. *The rank of the matrix M will increase within p steps. In other words, consider the matrix at any time such that* $\operatorname{rank}(M(t, t+k)) < p$, *then* $\operatorname{rank}(M(t, t+k+p)) \geq \operatorname{rank}(M(t, t+k)) + 1$.

Theorem V.4. *Algorithm 1 is complete and thus it produces a uniformly detectable sequence.*

Proof. By Lemma V.3, the maximum size of the M matrix is $p^2 - p + 1$. To show the sequence is uniformly detectable, we can follow an argument similar to the proof of Theorem IV.2. This can be accomplished by taking $s = z + 2p^2$ and examining the structure of $\tilde{B}(t, t+s)$ (note that the matrix M in the algorithm corresponds to sections of $B_{su}(t, t+s)$). The full proof can be found in [17]. □

Remark V.5 (Complexity of algorithm). The DETECTABLEGREEDY algorithm is a greedy algorithm with two extra steps: check which of the measurements increase rank, and reduce the M matrix. Following the analysis in [12] for the greedy part, and noting that finding the rank of a matrix is a (better then) $O(mn^4)$ operation [18], the total complexity is therefore $O(T(mn^2 + n^3 + mn^2 + mn^4))$. If we assume that in practice M will have $O(n)$ rows, the complexity of the algorithm becomes $O(Tmn^3)$. •

Algorithm 1: DETECTABLEGREEDY

Input: F: value function, (A, C, W, V): system parameters, T: time horizon.
Output: Sensor schedule with one sensor per time step.

```
A_su := [A_os 0; 0 A_ou] and C_su := [C_2 C_3] (cf. (7)).
p ← rank(A_su)
M ← 0
for t = 1...T do
    // Use (A_su, C_su) for constructing M
    foreach row r of C_su do
        if Appending row r multiplied by the proper power of
           A_su increases rank(M) then
            Mark r as valid.
    // Use (A, C) for selection
    if None of the rows are valid then Greedily select the
       best row of C.
    else Greedily select the best valid row from C.
    Update M with new measurement.
    if rank(M) == p then M ← 0
```

Although this shows that the sequence will indeed result in a bounded error covariance, we do not know what this bound is or even if it will be better than the regular greedy approach. However, in simulation, the results of our complete algorithm appear promising.

VI. SIMULATIONS

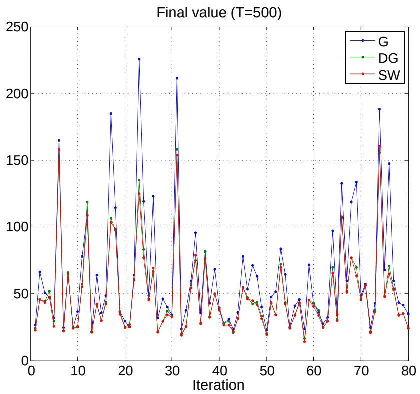
Here we present some simulations to investigate the properties of Algorithm 1 and how detectability impacts the bound for the error covariance. We also compare the algorithm to other known techniques. All the simulations are performed using the average of the trace of the *a priori* error covariance,

$$F(\sigma) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \operatorname{trace}(\Sigma_{t+1|t}),$$

For this section, we will refer to the GREEDY algorithm as G and the DETECTABLEGREEDY algorithm as DG.

In [6], an optimal algorithm to minimize the average trace of the covariance over a finite horizon is presented. This, however, is a time consuming operation. Therefore, we use a sliding window approximation for comparison. The SLIDINGWINDOW (SW) algorithm is basically an extended greedy such that the optimal is calculated via brute force over every nine time steps (this number was chosen merely to keep the runtime reasonable). This is repeated continuously until the desired time horizon is met. Note that although the optimal is achieved over the window size, the approximation may get worse the larger the time horizon.

We ran simulations for 80 randomly generated systems. All three algorithms were executed until $T = 500$ in order to allow the error covariance to settle to a steady value. We took



	1	2	3
G	4	4	72
DG	28	44	8
SW	48	32	0

Fig. 2. (Left) Final value of each algorithm over all iterations. (Right) Number of time each algorithm achieved each ranking.

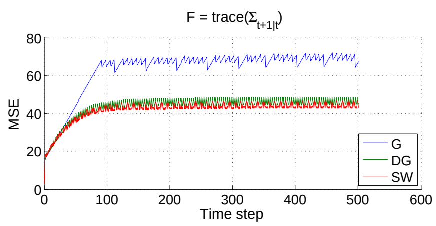


Fig. 3. A sample result. The value being plotted is the trace of the covariance (MSE) at each time step.

$n = m = 3$ and $A = I$. The C and W matrices had all their entries uniformly randomly distributed in $[0, 1]$ and $[0, 5]$ respectively. V was a diagonal matrix with the individual variances chosen uniformly randomly from $[0.5, 2]$.

Figure 2 shows the values of the final covariance for each algorithm as well as ranking counts of the algorithms. As we can see, 90% of the time DG algorithm outperforms the greedy. Although SW was the best in 60% of the instances, its output is very similar to DG.

Figure 3 shows a sample result for one of the iterations. The value being plotted is not the average trace but the trace of the covariance at every time step, i.e., the mean squared error (MSE). We can see that after an initial transition period, the covariance update becomes somewhat periodic. Figure 4 shows the amount of fluctuation in cost for each algorithm for each system once it has reached the steady state value. As we can see, on average all the algorithms have a similar amount of fluctuation, though DG was the best in terms of the worst-case and the spread of data.

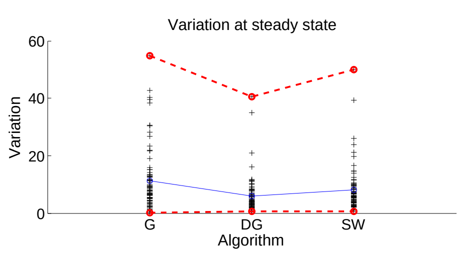


Fig. 4. The variation in the MSE after the covariance settles to a steady cycle for each system. The mean, max and min are also shown.

VII. Conclusions and Future Directions

We used the concept of uniform detectability to give necessary and sufficient conditions for the boundedness of error covariance resulting from a sensor schedule. We the presented a complete sensor scheduling algorithm; one which outputs a bounded sensor schedule if one exists. Finally, we showed that the error will die out exponentially over a large or infinite time horizon.

The DetectableGreedy algorithm outputs a sequence that is bounded; finding a way to determine the deviation from optimality (i.e., the actual bound) will help to quantify the performance. Also, investigating how to use the uniform detectability condition with other optimization techniques will help to determine the practicality of this approach.

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