ECE700.07: Game Theory with Engineering Applications

Lecture 3: Games in Normal Form

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Outline

- Strategic form games
- Dominant strategy equilibrium
- Pure and mixed Nash equilibrium
- Iterative elimination of strictly dominated strategies
- Price of anarchy
- Correlated equilibrium

- Readings:
  - MAS Sec. 3.2 and 3.4, GT Sec. 1 and 2

Strategic Form Games

- Agents act simultaneously without knowledge of others' actions
- Each game has to have
  - (1) Set of agents (2) Set of actions (3) Utilities
- Formally, strategic form game is triplet $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$
  - $\mathcal{I}$ is finite set of agents
  - S_i is set of available actions for agent i and $s_i \in S_i$ is action of agent i
  - $u_i : S \to \mathbb{R}$ is utility of agent i, where $S = \prod_i S_i$ is set of all action profiles
  - $s_{-i} = [s_j]_{j \neq i}$ is vector of actions for all agents except i
  - $S_{-i} = \prod_{j \neq i} S_j$ is set of all action profiles for all agents except i
  - $(s_i,\ s_{-i}) \in S$ is strategy profile, or outcome

Example: Prisoner's Dilemma

Prisoner 1 \ Prisoner 2	Stay Silent	Confess
Stay Silent	(-1, -1)	(-3, 0)
Confess	(0, -3)	(-2, -2)

- First number denotes utility of A1 and second number utility of A2
  - Row i and column j cell contains (x, y), where $x = u_1(i, j)$ and $y = u_2(i, j)$

Strategies

- Strategy is complete description of how to play
- It requires full contingent planning
  - As if you have to delegate play to "computer"
  - You would have to spell out how game should be played in every contingency
  - In chess, for example, this would be an impossible task
- In strategic form games, there is no difference between action and strategy (we will use them interchangeably)

Finite Strategy Spaces

- When S_i is finite for all i, game is called finite game
- For 2 agents and small action sets, it can be expressed in matrix form
- Example: matching pennies

Agent 1 \ Agent 2	Heads	Tails
Heads	(-1, 1)	(1, -1)
Tails	(1, -1)	(0, 0)

  - Game represents pure conflict; one player's utility is negative other player's utility; thus, zero sum game

Infinite Strategy Spaces

- When S_i is infinite for at least one i, game is called infinite game
- Example: Cournot competition
  - Two firms (agents) produce homogeneous good for same market
  - Agent i's action is quantity, $s_i \in [0, \infty]$, she produces
  - Agent i's utility is her total revenue minus total cost
    - $u_i(s_1, s_2) = s_i p(s_1 + s_2) - c s_i$
    - $p(s)$ is price as function of total quantity, c is unit cost (same for both agents)

Dominant Strategy

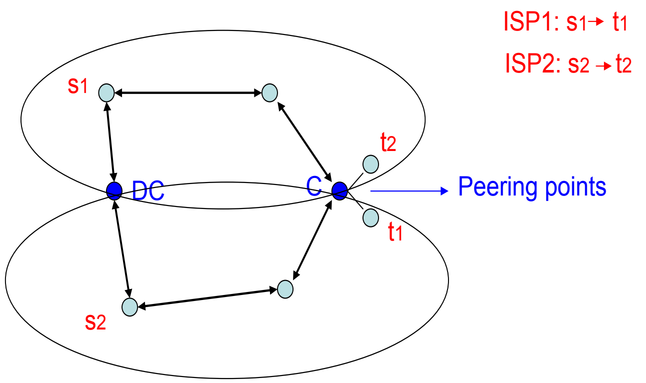
- Strategy $s_i \in S_i$ is dominant strategy for agent i if
$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i \text{ and for all } s_{-i} \in S_{-i}$$
- Example: prisoner's dilemma

Prisoner 1 \ Prisoner 2	Stay Silent	Confess
Stay Silent	(-1, -1)	(-3, 0)
Confess	(0, -3)	(-2, -2)

- Action "confess" strictly dominates action "stay silent"
- Self-interested, rational behavior does not lead to socially optimal result

Dominant Strategy Equilibrium

- Strategy profile s^* is (strictly) dominant strategy equilibrium if for each agent i, s_i^* is (strictly) dominant strategy
- Example: ISP routing game
  - ISPs share networks with other ISPs for free
  - ISPs choose to route traffic themselves or via partner
  - In this example, we assume cost along link is one



ISP 1 \ ISP 2	Route Yourself	Route via Partner
Route Yourself	(-3, -3)	(-6, -2)
Route via Partner	(-2, -6)	(-5, -5)

Dominated Strategies

- Strategy $s_i \in S_i$ is strictly dominated for agent i if $\exists s_i' \in S_i$:

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}),\ \forall\, s_{-i} \in S_{-i}$$

- Strategy $s_i \in S_i$ is weakly dominated for agent i if $\exists s_i' \in S_i$:

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),\ \forall\, s_{-i} \in S_{-i}$$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}),\ \exists\, s_{-i} \in S_{-i}$$

Rationality and Strictly Dominated Strategies

Prisoner 1 \ Prisoner 2	Stay Silent	Confess	Suicide
Stay Silent	(-1, -1)	(-3, 0)	(0, -10)
Confess	(0, -3)	(-2, -2)	(-1, -10)
Suicide	(-10, 0)	(-10, -1)	(-10, -10)

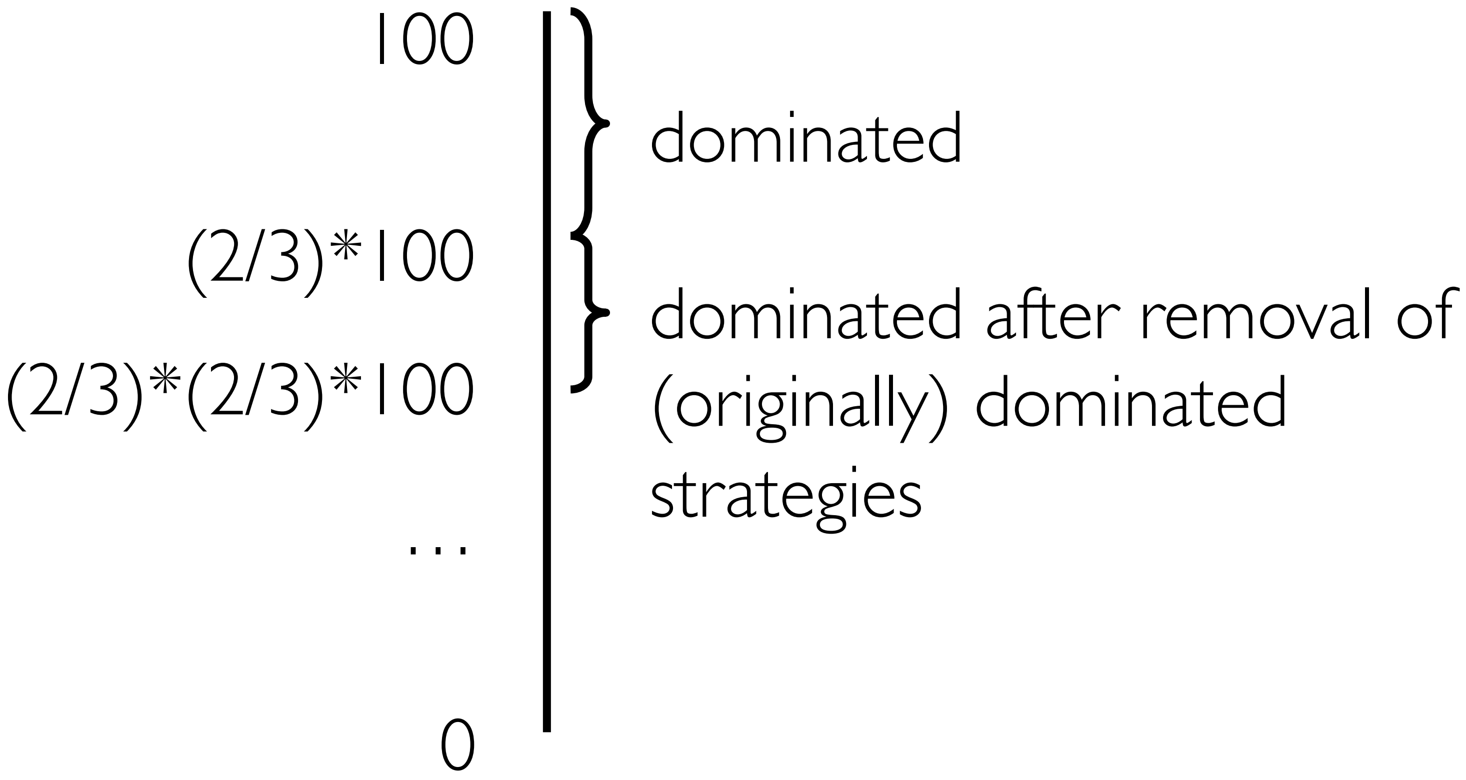
- There is no DS because of additional "suicide" strategy
  - Strictly dominated strategy for both prisoners
- No "rational" agent would choose "suicide"
  - No agent should play strictly dominated strategy

Rationality and Strictly Dominated Strategies (cont.)

- If A1 knows that A2 is rational, then she can eliminate A2's "suicide" strategy, and likewise for A2
- After one round of elimination of strictly dominated strategies, we are back to prisoner's dilemma game
- Iterated elimination of strictly dominated strategies leads to unique outcome, "confess, confess"
- Game is dominance solvable (We will come back to this later)

How Reasonable is Dominance Solvability?

- Consider k-beauty contest game is dominance solvable!



Existence of Dominant Strategy Equilibrium

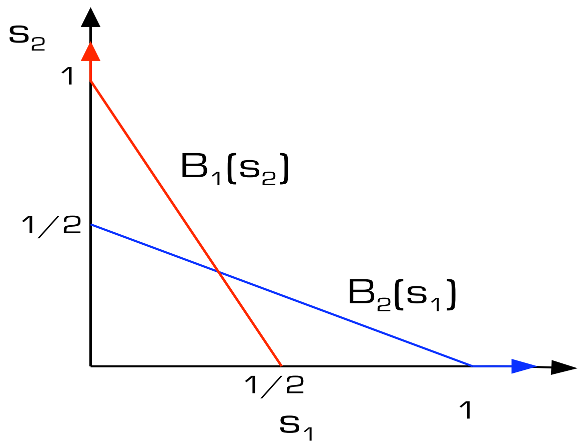
- Does matching pennies game have DSE?

Agent 2 / Agent 1	Heads	Tails
Heads	(-1, 1)	(1, -1)
Tails	(1, -1)	(-1, 1)

- Dominant strategy equilibria do not always exist

Best Response

- $B_i(s_{-i})$ represents agent i's best response correspondence to s_{-i}
- Example: Cournot competition
  - $u_i(s_1, s_2) = s_i p(s_1 + s_2) - c s_i$
  - Suppose that $c = 1$ and $p(s) = \max\{0, 2 - s\}$
  - First order optimality condition gives



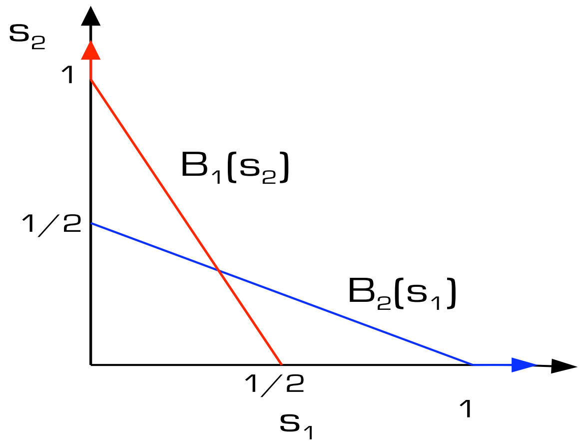
$$B_i(s_{-i}) = \arg\max(s_i(2 - s_i - s_{-i}) - s_i)$$

$$= \begin{cases} (1 - s_{-i})/2 & \text{if } s_{-i} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

  - Figure illustrates best response correspondences (functions here!)

Pure Strategy Nash Equilibrium

- (Pure strategy) Nash equilibrium is strategy profile $s^* \in S$ such that
$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \qquad \forall i, s_i \in S_i$$
- No agent can profitably deviate given strategies of others
- In Nash equilibrium, best response correspondences intersect
- Strategy profile $s^* \in S$ is Nash equilibrium iff $s_i^* \in B_i(s_{-i}^*), \forall i$



Example: Battle of the Sexes

Husband \ Wife	Football	Opera
Football	(4, 1)	(-1, -1)
Opera	(-1, -1)	(1, 4)

- Couple agreed to meet this evening
- They cannot recall if they will be attending opera or football
- Husband prefers football, wife prefers opera
- Both prefer to go to same place rather than different ones

Existence of Pure Strategy Nash Equilibrium

- Does matching pennies game have pure strategy NE?

Agent 1 \ Agent 2	Heads	Tails
Heads	(-1, 1)	(1, -1)
Tails	(1, -1)	(-1, 1)

- Pure strategy Nash equilibria do not always exist

Mixed Strategies

- Let Σ_i denote set of probability measures over pure strategy set S_i
  - E.g., 45% left, 10% middle, and 45% right
- We use $\sigma_i \in \Sigma_i$ to denote mixed strategy of agent i, and $\sigma \in \Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$ to denote mixed strategy profile
  - This implicitly assumes agents randomize independently
- Similarly, we define $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$
- Following von Neumann-Morgenstern expected utility theory, we have

$$u_i(\sigma) = \int_S u_i(s) d\sigma(s)$$

Strict Dominance by Mixed Strategy

Agent 2 / Agent 1	a	b
a	(2, 0)	(-1, 0)
b	(0, 0)	(0, 0)
c	(-1, 0)	(2, 0)

- Agent 1 has no pure strategy that strictly dominates b
- However, b is strictly dominated by mixed strategy $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$
- Action s_i is strictly dominated if there exists σ_i such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}), \quad \forall s_{-i} \in S_{-i}$
- Strictly dominated strategy is never played with positive probability in mixed strategy NE
- However, weakly dominated strategies could be used in Nash equilibrium

Iterative Elimination of Strictly Dominated Strategies

- Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$
- For each agent i, define
  - $S_i^n = \{s_i \in S_i^{n-1} \mid \nexists \sigma_i \in \Sigma_i^{n-1}: \quad u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}^{n-1}\}$
- And define
  - $\Sigma_i^n = \{\sigma_i \in \Sigma_i | \sigma_i(s_i) > 0$ only if $s_i \in S_i^n\}$
- Finally, define S_i^∞ as set of agent i's strategies that survive IESDS
  - $S_i^\infty = \cap_{n=1}^\infty S_i^n$

Mixed Strategy Nash Equilibrium

- Profile σ^* is (mixed strategy) Nash equilibrium if for each agent i

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \qquad \forall \sigma_i \in \Sigma_i$$

- Profile σ^* is (mixed strategy) Nash equilibrium iff for each agent i

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*), \qquad \forall s_i \in \Sigma_i$$

  - Why?
  - Hint: Agent i's utility for playing mix strategies is convex combination of his utility when playing pure strategies

Mixed Strategy Nash Equilibria (cont.)

- For G, finite strategic form game, profile σ^* is NE iff for each agent, every pure strategy in support of σ_i^* is best response to σ_{-i}^*
  - Why?
  - Hint: If profile σ^* puts positive probability on strategy that is not best response, shifting that probability to other strategies improves expected utility
- Every action in support of agent's NE mixed strategy yields same utility

Finding Mixed Strategy Nash Equilibrium

Husband \ Wife	Football	Opera
Football	(4, 1)	(-1, -1)
Opera	(-1, -1)	(1, 4)

- Assume H goes to football with probability p and W goes to opera with probability q
- Using mixed equilibrium characterization, we have

$$p - (1 - p) = -p + 4(1 - p) \Rightarrow p = \frac{5}{7}$$

$$q - (1 - q) = -q + 4(1 - q) \Rightarrow q = \frac{5}{7}$$

- Mixed strategy Nash equilibrium utilities are $\left(\frac{3}{7}, \frac{3}{7}\right)$

Example: Bertrand Competition with Capacity Constraints

- Two firms charge prices $p_1, p_2 \in [0, 1]$ per unit of same good
- There is unit demand which has to be supplied
- Customers prefer firm with lower price
- Assume each firm has capacity constraint of 2/3 units of demand
  - If $p_1 < p_2$, firm 2 gets 1/3 units of demand
- If both firms charge same price, each gets half of demand
- Utility of each firm is profit they make ($c = 0$, for both firms)

Example: Bertrand Competition with Capacity Constraints (cont.)

- Without capacity constraint, $p_1 = p_2 = 0$ is unique pure strategy NE
  - You will prove this in first assignment!
- With capacity constraint, $p_1 = p_2 = 0$ is no longer pure strategy NE
  - Either firm can increase its price and still have 1/3 units of demand
- We consider symmetric mixed strategy Nash equilibrium
  - I.e., both firms use same mixed strategy
- We use cumulative distribution function, $F(\cdot)$, for mixed strategies

Example: Bertrand Competition with Capacity Constraints (cont.)

- What is expected utility of firm 1 when it chooses p_1 and firm 2 uses mixed strategy $F(\cdot)$?

$$u_1(p_1, F(\cdot)) = F(p_1)\frac{p_1}{3} + \left(1 - F(p_1)\right)\frac{2p_1}{3}$$

- Each action in support of mixed strategy must yield same utility at NE
  - $\forall\, p$ in support of $F(\cdot)$

$$\frac{2p}{3} - F(p)\frac{p}{3} = k,$$

  - $\exists\, k \geq 0$

$$F(p) = 2 - \frac{3k}{p}$$

Example: Bertrand Competition with Capacity Constraints (cont.)

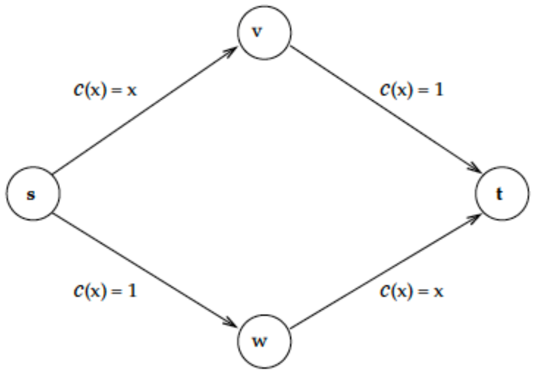
- Note that upper support of mixed strategy must be at $p = 1$, which implies that $F(1) = 1$
- Combining with preceding, we obtain

$$F(p) = \begin{cases} 0, & \text{if } 0 \le p \le \frac{1}{2} \\ 2 - \frac{1}{p}, & \text{if } \frac{1}{2} \le p \le 1 \\ 1, & \text{if } p \ge 1. \end{cases}$$

Nash's Theorem

- Theorem (Nash): Every finite game has mixed strategy NE
- Why is this important?
  - Without knowing the existence of equilibrium, it is difficult (perhaps meaningless) to try to understand its properties
  - Armed with this theorem, we also know that every finite game has at least one equilibrium, and thus we can simply try to locate equilibria
  - Knowing that there might be multiple equilibria, we should study efficiency/inefficiency of games' equilibria

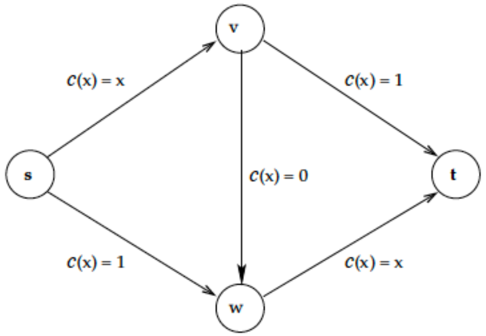
Example: Braess's Paradox



- There are $2k$ drivers commuting from s to t
- $C(x)$ indicates travel time in hours for x drivers
- k drivers going through v and k going through w is NE
  - Why?

T. Roughgarden, Lectures Notes on Algorithmic Game Theory

Example: Braess's Paradox (cont.)



- Suppose we install teleportation device allowing drivers to travel instantly from v to w
  - What is new NE? What is drivers' commute time?
  - What is optimal commute time?
  - Does selfish routing does not minimize commute time?
- Price of Anarchy (PoA) is ratio between system performance with strategic agents and best possible system performance
  - Ratio between 2 and 3/2 in Braess's Paradox

Correlated Strategies

- In NE, agents randomize over strategies independently
- Agents can randomize by communicating prior to taking actions
- Example: battle of the sexes

Husband \ Wife	Football	Opera
Football	(4, 1)	(-1, -1)
Opera	(-1, -1)	(1, 4)

  - Unique mixed strategy NE is $\left(\left(\frac{5}{7}, \frac{2}{7}\right), \left(\frac{5}{7}, \frac{2}{7}\right)\right)$ with utilities $\left(\frac{3}{7}, \frac{3}{7}\right)$
  - Can they both do better by coordinating?

Correlated Strategies (cont.)

- Suppose there is publicly observable fair coin
- If it is heads/tails, they both get signal to go to football/opera
- If H/W sees heads, he/she believes that W/H will go to football, and therefore going to football is his/her best response
  - Similar argument can be made when he/she sees tails
- When recommendation of coin is part of Nash equilibrium, no agent has any incentives to deviate
- Expected utilities for this play of game increases to $(2.5, 2.5)$

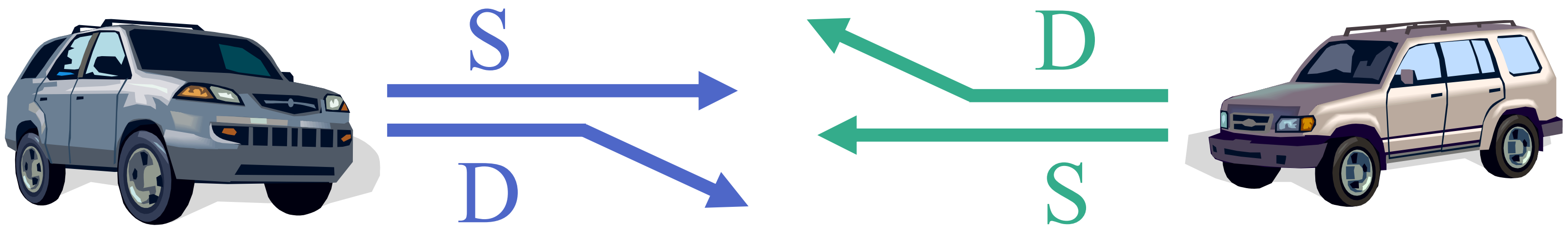
Correlated Equilibrium

- Correlated equilibrium of finite game is joint probability distribution $\pi \in \Delta(S)$ such that $\forall\ i, s_i \in S_i$ with $\pi(s_i) > 0$, and $t_i \in S_i$

$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i} | s_i)[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})] \geq 0$$

- Distribution π is defined to be correlated equilibrium if no agent can benefit by deviating from her recommendation, assuming other agents play according to their recommendations

Example: Game of Chicken



Driver 2 Driver 1	S	D
S	(-5, -5)	(1, -1)
D	(-1, 1)	(0, 0)

- (D, S) and (S, D) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing strategy will make agents strictly worse off

Example: Game of Chicken (cont.)

Driver 1 \ Driver 2	S	D
S	(-5, -5)	(1, -1)
D	(-1, 1)	(0, 0)

- Assume D1 dodges with probability p and D2 dodges with probability q
- Using mixed equilibrium characterization, we have

$$p - 5(1 - p) = 0 - (1 - p) \Rightarrow p = \frac{4}{5}$$

$$q - 5(1 - q) = 0 - (1 - q) \Rightarrow q = \frac{4}{5}$$

- Mixed strategy Nash equilibrium utilities are $\left(\frac{-1}{5}, \frac{-1}{5}\right)$, people may die!

Example: Game of Chicken (cont.)

Driver 2 / Driver 1	S	D
S	(-5, -5) 0%	(1, -1) 40%
D	(-1, 1) 40%	(0, 0) 20%

- Is this correlated equilibrium?
- If D1 gets signal to dodge
  - Conditional probability that D2 dodges is $\frac{0.2}{0.2+0.4} = \frac{1}{3}$
  - Expected utility of dodging is $\left(\frac{2}{3}\right)\times(-1)$
  - Expected utility of going straight is $\left(\frac{1}{3}\right)\times 1 + \left(\frac{2}{3}\right)\times(-5) = -3$
  - Following recommendation is better
- If D1 gets signal to go straight, she knows that D2 is told to dodge, so again, D1 wants to follow recommendation
- Similar analysis works for D2, so nobody dies!
- Expected utilities increase to $(0, 0)$

Characterization of Correlated Equilibrium

- Proposition
  - Joint distribution $\pi \in \Delta(S)$ is correlated equilibrium of finite game iff

$$\sum_{s_{-i} \in S_{-i}} \pi(s)[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})] \geq 0, \qquad \forall i, s_i, t_i \in S_i$$

- Proof
  - By definition of conditional probability, correlated equilibrium can be written as

$$\sum_{s_{-i} \in S_{-i}} \frac{\pi(s_i, s_{-i})}{\sum_{t_{-i} \in S_{-i}} \pi(s_i, t_{-i})} [u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})] \geq 0, \forall i, s_i \in S_i \text{ with } \pi(s_i) > 0, \text{ and } t_i$$

  - Denominator does not depend on variable of sum, so it can be factored and cancelled
  - If $\pi(s_i) = 0$, then LHS of Proposition is zero regardless of i and t_i, so equation always holds

Questions?

Acknowledgement

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