ECE700.07: Game Theory with Engineering Applications

Lecture 4: Computing Solution Concepts of Normal Form Games

Seyed Majid Zahedi

UNIVERSITY OF WATERLOO

Outline

- Brief overview of (mixed integer) linear programs
- Solving for
  - Dominated strategies
  - Minimax and maximin strategies
  - Nash equilibrium
  - Correlated NE

- Readings:
  - MAS Appendix B, and Sec. 4

Linear Program Example: Reproduction of Two Paintings

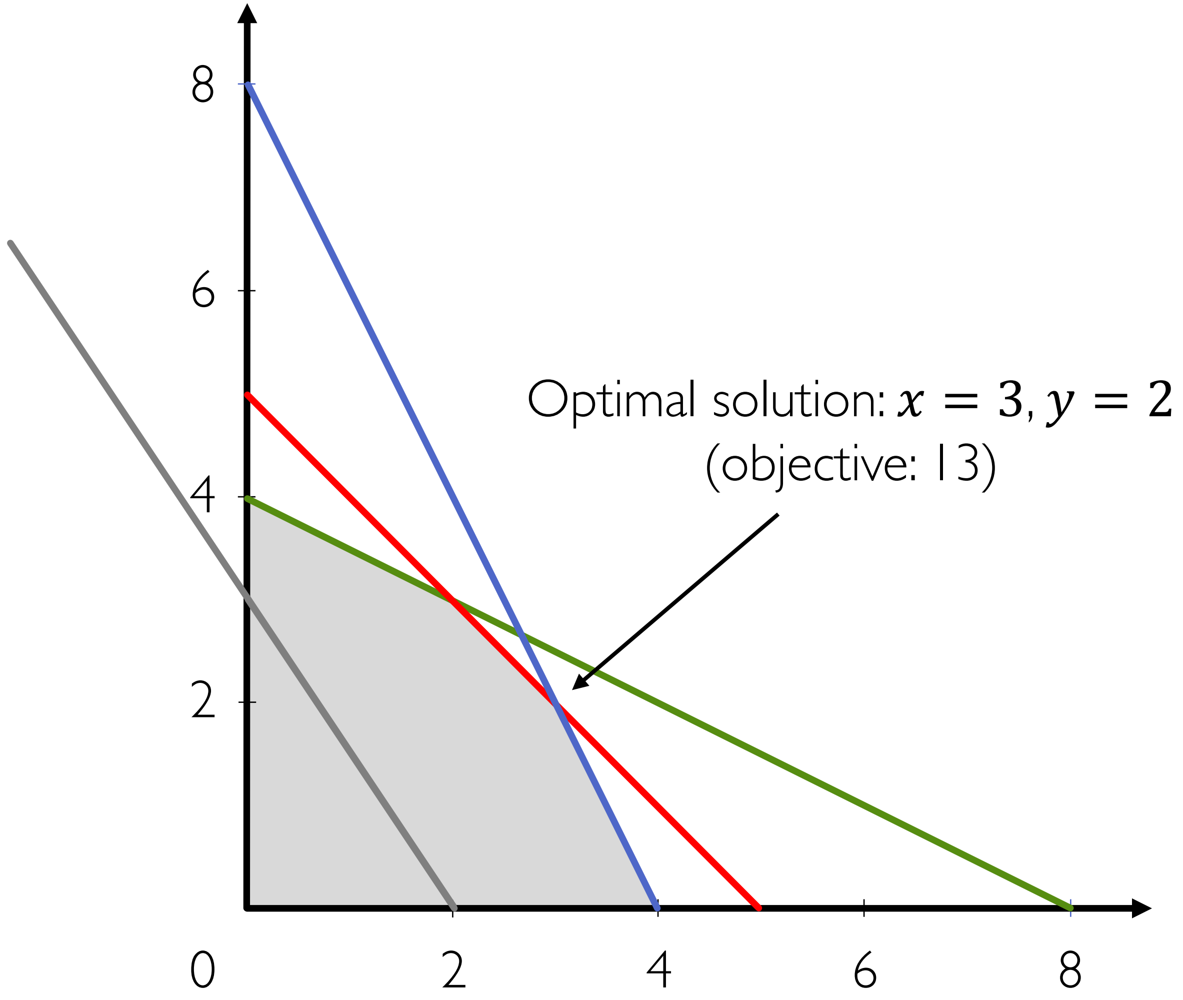
 

- Painting 1 sells for \$30
- Painting 2 sells for \$20
- We have 16 units of blue, 8 green, 5 red
- Painting 1 requires 4 blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red

$$
\begin{aligned}
\text{max.} \quad & 3x + 2y \\
\text{s.t.} \quad & 4x + 2y \leq 16 \\
& x + 2y \leq 8 \\
& x + y \leq 5 \\
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$

Solving Linear Program Graphically

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & 4x + 2y \le 16 \\ & x + 2y \le 8 \\ & x + y \le 5 \\ & x \ge 0 \\ & y \ge 0 \end{aligned}$$



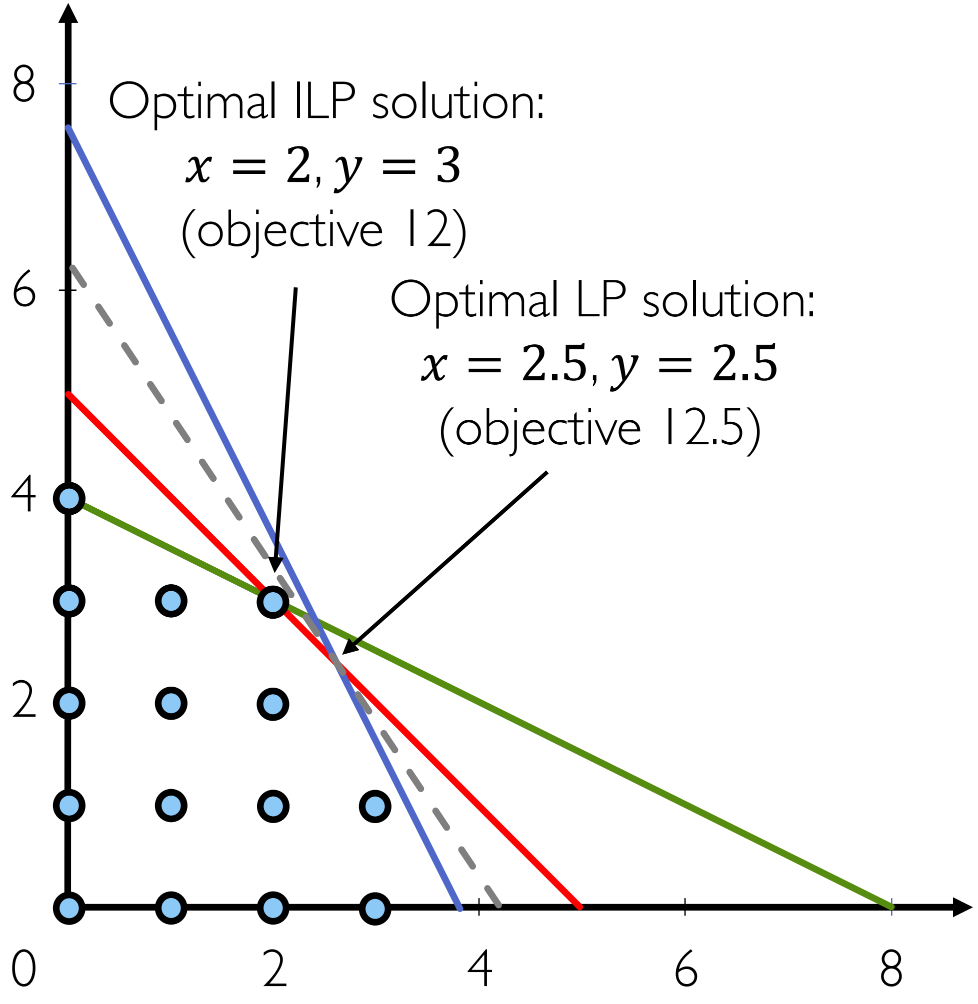
Modified LP

$$
\begin{aligned}
\text{max.} \quad & 3x + 2y \\
\text{s.t.} \quad & 4x + 2y \leq 15 \\
& x + 2y \leq 8 \\
& x + y \leq 5 \\
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$

- Optimal solution: x = 2.5, y = 2.5
- Objective = 7.5 + 5 = 12.5
- Can we sell half paintings?

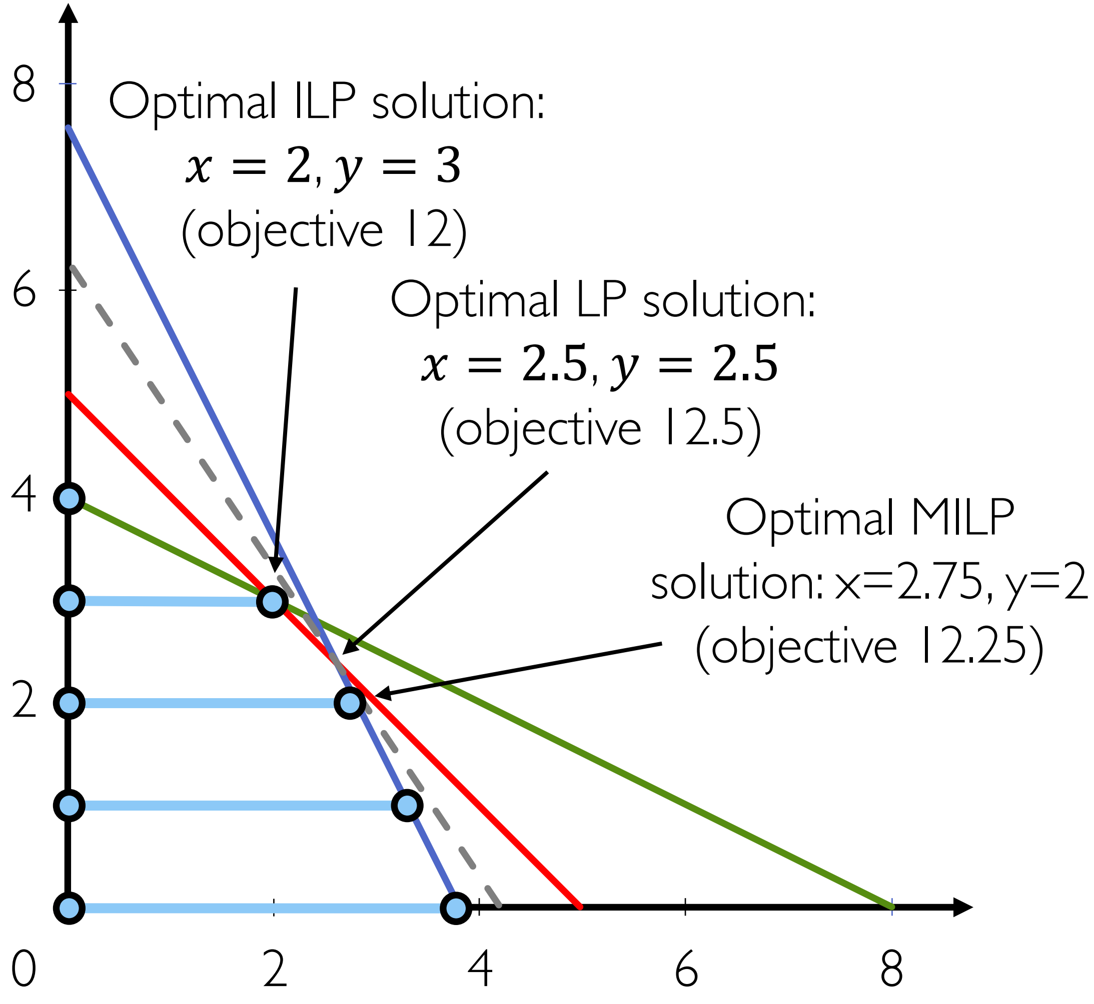
Integer Linear Program

$$\begin{aligned} \text{max.} \quad & 3x + 2y \\ \text{s.t.} \quad & 4x + 2y \leq 15 \\ & x + 2y \leq 8 \\ & x + y \leq 5 \\ & x \in \mathbb{N}_0 \\ & y \in \mathbb{N}_0 \end{aligned}$$



Mixed Integer Linear Program

$$
\begin{aligned}
\text{max.} \quad & 3x + 2y \\
\text{s.t.} \quad & \color{blue}{4x + 2y \leq 15} \\
& \color{green}{x + 2y \leq 8} \\
& \color{red}{x + y \leq 5} \\
& x \geq 0 \\
& y \in \mathbb{N}_0
\end{aligned}
$$



Solving Mixed Linear/Integer Programs

- Linear programs can be solved efficiently
  - Simplex, ellipsoid, interior point methods, etc.
- (Mixed) integer programs are NP-hard to solve
  - Many standard NP-complete problems can be modelled as MILP
  - Search type algorithms such as branch and bound
- Standard packages for solving these
  - Gurobi, MOSEK, GNU Linear Programming Kit, CPLEX, CVXPY, etc.
- LP relaxation of (M)ILP: remove integrality constraints
  - Gives upper bound on MILP (~admissible heuristic)

Exercise I in Modeling: Knapsack-type Problem

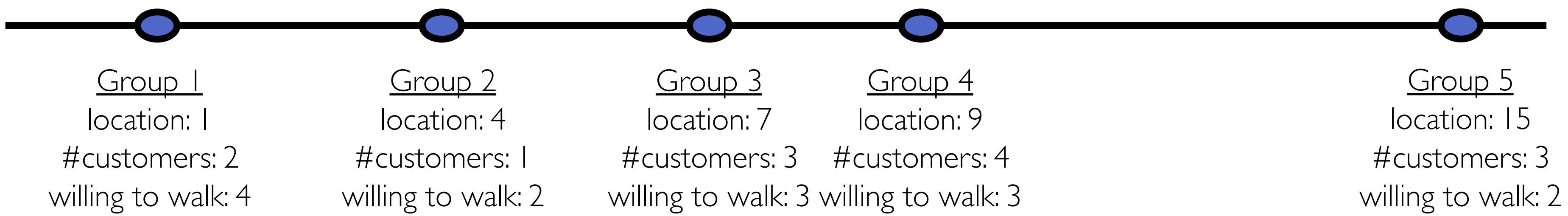
- We arrive in room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for $11 (3 units available)
- Unit of object B: 4kg, 4 liters, sells for $4 (4 units available)
- Unit of object C: 6kg, 3 liters, sells for $9 (1 unit available)
- What should we take?

Exercise II in Modeling: Cell Phones (Set Cover)

- We want to have a working phone in every continent (besides Antarctica) but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E

Exercise III in Modeling: Hot-dog Stands

- We have two hot-dog stands to be placed in somewhere along beach
- We know where groups of people who like hot-dogs are
- We also know how far each group is willing to walk
- Where do we put our stands to maximize #hot-dogs sold? (price is fixed)



Checking for Strict Dominance by Mixed Strategies

- LP for checking if strategy t_i is strictly dominated by any mixed strategy

$$
\begin{aligned}
\text{max.} \quad & \epsilon \\
\text{s.t.} \quad & \sum_{s_i \in S_i} p_{s_i} u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i}) + \epsilon, \quad && \forall s_{-i} \in S_{-i} \\
& \sum_{s_i \in S_i} p_{s_i} = 1 \\
& p_{s_i} \geq 0, && \forall s_i \in S_i
\end{aligned}
$$

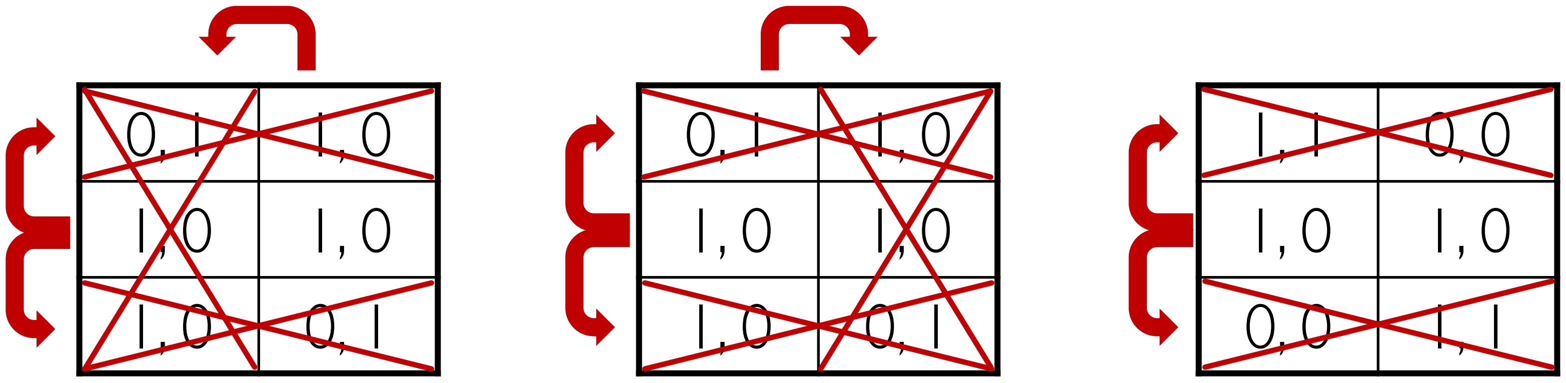
Checking for Weak Dominance by Mixed Strategies

- LP for checking if strategy t_i is weakly dominated by any mixed strategy

$$
\begin{aligned}
\text{max.} \quad & \sum_{s_{-i} \in S_{-i}} \left(\left(\sum_{s_i \in S_i} p_{s_i} u_i(s_i, s_{-i}) \right) - u_i(t_i, s_{-i}) \right) & \\
\text{s.t.} \quad & \sum_{s_i \in S_i} p_{s_i} u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i}), & \forall s_{-i} \in S_{-i} \\
& \sum_{s_i \in S_i} p_{s_i} = 1 & \\
& p_{s_i} \geq 0, & \forall s_i \in S_i
\end{aligned}
$$

Path Dependency of Iterated Dominance

- Iterated weak dominance is path-dependent
  - Sequence of eliminations may determine which solution we get (if any)



- Iterated strict dominance is path-independent:
  - Elimination process will always terminate at the same point

Two Computational Questions for Iterated Dominance

- 1. Can any given strategy be eliminated using iterated dominance?
- 2. Is there some path of elimination by iterated dominance such that only one strategy per player remains?
- For strict dominance (with or without dominance by mixed strategies), both can be solved in polynomial time due to path-independence
  - Check if any strategy is dominated, remove it, repeat
- For weak dominance, both questions are NP-hard (even when all utilities are 0 or 1), with or without dominance by mixed strategies [Conitzer, Sandholm 05], and weaker version proved by [Gilboa, Kalai, Zemel 93]

Minimax and Maximin Values

- Maximin strategy for agent i (leading to maximin value for agent i)

$$\arg\max_{\sigma_i} \min_{s_{-i}} u_i(\sigma_i, s_{-i})$$

- Minimax strategy of other agents (leading to minimax value for agent i)

$$\arg\min_{\sigma_{-i}} \max_{s_i} u_i(s_i, \sigma_{-i})$$

LP for Calculating Maximin Strategy and Value

$$
\begin{aligned}
\text{max.} \quad & u \\
\text{s.t.} \quad & \sum_{s_i \in S_i} p_{s_i} u_i(s_i, s_{-i}) \geq u, \quad && \forall s_{-i} \in S_{-i} \\
& \sum_{s_i \in S_i} p_{s_i} = 1 \\
& p_{s_i} \geq 0, && \forall s_i \in S_i
\end{aligned}
$$

- Objective of this LP, u, is maximin value of agent i
- Given p_{s_i}, first constraint ensures that u is less than any achievable expected utility for any pure strategies of opponents

Minimax Theorem [von Neumann 1928]

- Each player's NE utility in any finite, two-player, zero-sum game is equal to her maximin value and minimax value

$$\max_{\sigma_i} \min_{s_{-i}} u_i(\sigma_i, s_{-i}) = \min_{\sigma_{-i}} \max_{s_i} u_i(s_i, \sigma_{-i})$$

- Minimax theorem does not hold with pure strategies only (example?)

Example

Agent 2 / Agent 1	Left	Right
Up	(20, -20)	(0, 0)
Down	(0, 0)	(10, -10)

- What is maximin value of agent 1 with and without mixed strategies?
- What is minimax value of agent 1 with and without mixed strategies?
- What is NE of this game?

Solving NE of Two-Player, Zero-Sum Games

- Minimax value of agent 1

$$
\begin{aligned}
\text{min.} \quad & u_1 \\
\text{s.t.} \quad & \sum_{s_2 \in S_2} p_{s_2} u_1(s_1, s_2) \leq u_1 && \forall s_1 \in S_1 \\
& \sum_{s_2 \in S_2} p_{s_2} = 1 \\
& p_{s_2} \geq 0, && \forall s_2 \in S_2
\end{aligned}
$$

- Maximin value of agent 1

$$
\begin{aligned}
\text{max.} \quad & u_1 \\
\text{s.t.} \quad & \sum_{s_1 \in S_1} p_{s_1} u_1(s_1, s_2) \geq u_1 && \forall s_2 \in S_2 \\
& \sum_{s_1 \in S_1} p_{s_1} = 1 \\
& p_{s_1} \geq 0, && \forall s_1 \in S_1
\end{aligned}
$$

- NE is expressed as LP, which means equilibria can be computed in polynomial time

Maximin Strategy for General-Sum Games

- Agents could still play minimax strategy in general-sum games
  - I.e., pretend that the opponent is only trying to hurt you
- But this is not rational:

Agent 1 \ Agent 2	Left	Right
Up	(0, 0)	(3, 1)
Down	(1, 0)	(2, 1)

  - If A2 was trying to hurt A1, she would play Left, so A1 should play Down
  - In reality, A2 will play Right (strictly dominant), so A1 should play Up

Hardness of Computing NE for General-Sum Games

- Complexity was open for long time
  - "together with factoring […] the most important concrete open question on the boundary of P today" [Papadimitriou STOC'01]
- Sequence of papers showed that computing any NE is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require exponential time (in worst case)

Hardness of Computing NE for General-Sum Games (cont.)

- What about computing NE with specific property?
  - NE that is not Pareto-dominated
  - NE that maximizes expected social welfare (i.e., sum of all agents' utilities)
  - NE that maximizes expected utility of given agent
  - NE that maximizes expected utility of worst-off player
  - NE in which given pure strategy is played with positive probability
  - NE in which given pure strategy is played with zero probability
  - …
- All of these are NP-hard (and the optimization questions are inapproximable assuming P != NP), even in 2-player games
  [Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]

Search-Based Approaches (for Two-Player Games)

- We can use (feasibility) LP, if we know support X_i of each player i's mixed strategy
  - I.e., we know which pure strategies receive positive probability

$$
\begin{aligned}
\text{find} \quad & (u_1, u_2) \\
\text{s.t.} \quad & p_{s_i} \geq 0, && \forall i, s_i \in S_i \\
& \sum_{s_i \in S_i} p_{s_i} = 1, && \forall i \\
& p_{s_i} = 0, && \forall i, s_i \in S_i / X_i \\
& \sum_{s_{-i} \in S_{-i}} p_{s_{-i}} u_i(s_i, s_{-i}) = u_i, && \forall i, s_i \in X_i \\
& \sum_{s_{-i} \in S_{-i}} p_{s_{-i}} u_i(s_i, s_{-i}) \leq u_i, && \forall i, s_i \in S_i / X_i
\end{aligned}
$$

- Thus, we can search over possible supports, which is basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GEB08]

Solving for NE using MILP (for Two-Player Games)

[Sandholm, Gilpin, Conitzer AAAI05]

$$
\begin{array}{lll}
\text{max.} & \text{whatever you like (e.g., social welfare)} & \\
\text{s.t.} & p_{s_i} \geq 0, & \forall i, s_i \in S_i \\
& \displaystyle\sum_{s_i \in S_i} p_{s_i} = 1, & \forall i \\
& \displaystyle\sum_{s_{-i} \in S_{-i}} p_{s_{-i}} u_i(s_i, s_{-i}) = u_{s_i}, & \forall i, s_i \in S_i \\
& u_{s_i} \leq u_i, & \forall i, s_i \in S_i \\
& p_{s_i} \leq b_{s_i}, & \forall i, s_i \in S_i \\
& u_i - u_{s_i} \leq M(1 - b_{s_i}), & \forall i, s_i \in S_i \\
& b_{s_i} \in \{0, 1\}, & \forall i, s_i \in S_i
\end{array}
$$

- b_{s_i} is binary variable indicating if s_i is in support of i's mixed strategy, and M is large number

Solving for Correlated Equilibrium using LP (N-Player Games!)

- Variables are now p_s where s is profile of pure strategies (i.e., outcome)

$$
\begin{aligned}
\text{max.} \quad & \text{whatever you like (e.g., social welfare)} \\
\text{s.t.} \quad & \sum_{s_{-i} \in S_{-i}} p_s u_i(s) \geq \sum_{s_{-i} \in S_{-i}} p_s u_i(t_i, s_{-i}) \qquad \forall i, s_i \in S_i, t_i \in S_i \\
& \sum_{s \in S} p_s = 1 \\
& p_s \geq 0, \qquad \forall s \in S
\end{aligned}
$$

Questions?

Acknowledgement

- This lecture is a slightly modified version of ones prepared by
  - Vincent Conitzer [Duke CPS 590.4]