ECE700.07: Game Theory with Engineering Applications

Lecture 5: Games in Extensive Form

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Outline

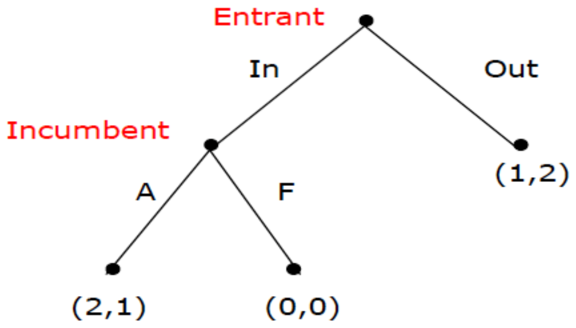
- Perfect information extensive form games
- Subgame perfect equilibrium
- Backward induction
- One-shot deviation principle
- Imperfect information extensive form games

- Readings:
  - MAS Sec. 5, GT Sec. 3 (skim through Sec. 3.4 and 3.6), Sec. 4.1, and Sec 4.2

Extensive Form Games

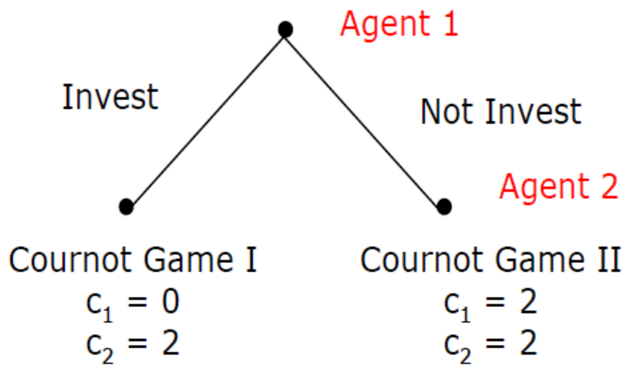
- So far, we have studied strategic form games
  - Agents take actions once and simultaneously
- Next, we study extensive form games
  - Agents sequentially make decisions in multi-stage games
  - Some agents may move simultaneously at some stage
  - Extensive form games can be conveniently represented by game trees

Example: Entry Deterrence Game



- Entrant chooses to enter market or stay out
- Incumbent, after observing entrant's action, chooses to accommodate or fight
- Utilities are given by (x, y) at leaves for each action profile (or history)
  - x denotes utility of agent 1 (entrant) and y denotes utility of agent 2 (incumbent)

Example: Investment in Duopoly



- Agent 1 chooses to invest or not invest
- After that, both agents engage in Cournot competition
  - If agent 1 invests, then they engage in Cournot game with $c_1 = 0$ and $c_2 = 2$
  - Otherwise, they engage in Cournot game with $c_1 = c_2 = 2$

Finite Perfect-Information Extensive Form Games

- Formally, each game is tuple $G = \langle \mathcal{I}, (\mathcal{A}_i)_{i \in \mathcal{I}}, \mathcal{H}, \mathcal{Z}, \alpha, (\beta_i)_{i \in \mathcal{I}}, \rho, (u_i)_{i \in \mathcal{I}} \rangle$
  - $\mathcal{I}$ is finite set of agents
  - $\mathcal{A}_\mathrm{i}$ is set of actions available to agent $\boldsymbol{i}$
  - $\mathcal{H}$ is set of choice nodes (internal nodes of game tree)
  - $\mathcal{Z}$ is set of terminal nodes (leaves of game tree)
  - $\alpha: \mathcal{H} \mapsto 2^{\mathcal{I}}$ is agent function, which assigns to each choice node set of agents
  - $\beta_\mathrm{i}: \mathcal{H} \mapsto 2^{\mathcal{A}_i}$ is action function, which maps choice nodes to set of actions available to agent $\boldsymbol{i}$
  - $\rho: \mathcal{H} \times \mathcal{A} \mapsto \mathcal{H} \cup \mathcal{Z}$ is successor function, which maps choice nodes and action profiles to new choice or terminal node, such that if $\rho(h_1, a_1) = \rho(h_2, a_2)$, then $h_1 = h_2$ and $a_1 = a_2$
  - $u_i: \mathcal{Z} \mapsto \mathbb{R}$ is utility function, which assigns real-valued utility to agent $\boldsymbol{i}$ at terminal nodes

History in Extensive Form Games

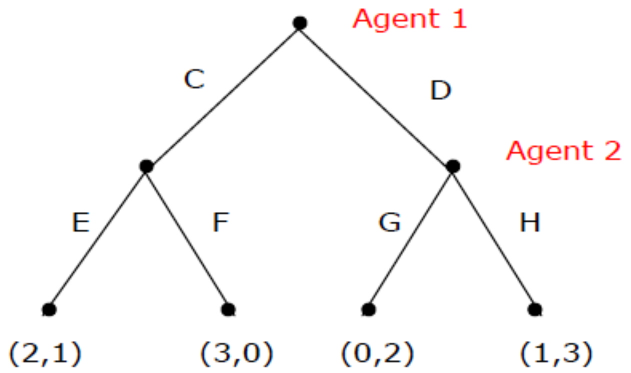
- Let $H^k = \{h^k\} \subseteq \mathcal{H} \cup \mathcal{Z}$ be set of all possible stage k nodes in game's tree
  - $h^0 = \emptyset$ — initial history
  - $a^0 = \left(a_i^0\right)_{i\in\alpha(h^0)}$ — stage 0 action profile
  - $h^1 = (a^0)$ — *history* after stage 0
  - $a^1 = \left(a_i^1\right)_{i\in\alpha(h^1)}$ — stage 1 action profile
  - $h^2 = (a^0, a^1)$ — *history* after stage 1
  - $\vdots$ — $\vdots$
  - $h^k = \left(a^0, \dots, a^{k-1}\right)$ — *history* after stage $k-1$
- If number of stages is finite, then game is called finite horizon game
- In perfect information extensive form games, each choice (and terminal) node is associated with unique history and <u>vice versa</u>

Strategies in Extensive Form Games

- Pure strategies for agent $\boldsymbol{i}$ is defined as contingency plan for every choice node that agent $\boldsymbol{i}$ is assigned to

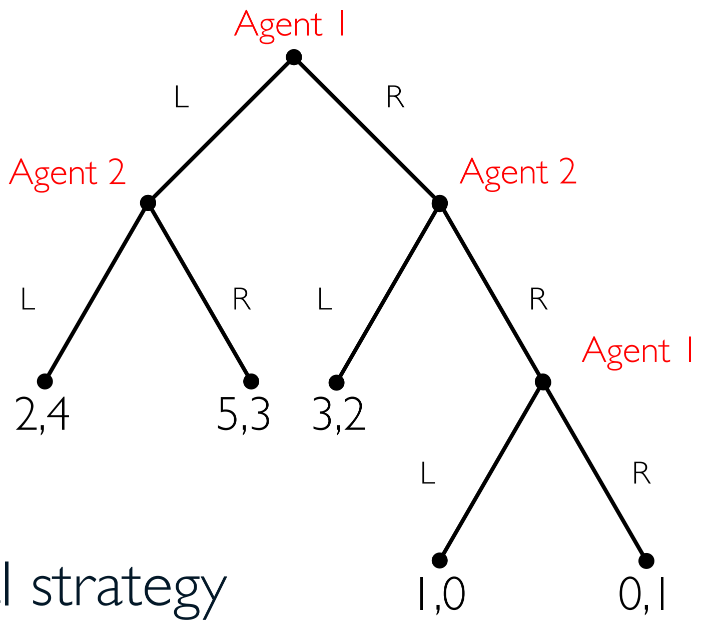
$$s_i \in S_i = \prod_{h \in \mathcal{H}: i \in \alpha(h)} \beta_{i(h)}$$

- Example:
  - Agent 1's strategies: $s_1 \in S_1 = \{C, D\}$
  - Agent 2's strategies: $s_2 \in S_2 = \{EG, EH, FG, \mathrm{FH}\}$
  - For strategy profile $s = (C, EG)$, outcome is terminal node $\{C, E\}$



Randomized Strategies in Extensive Form Games

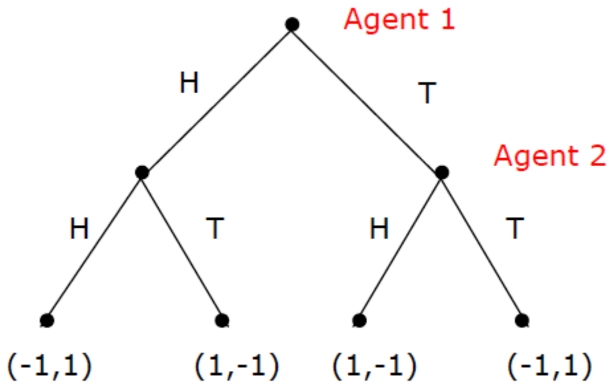
- Mixed strategy: randomizing over pure strategies
- Behavioral strategy: randomizing at each choice node



- Example:
  - Give behavioral strategy for agent 1
    - L with probability 0.2 and L with probability 0.5
  - Give mixed strategy for agent 1 that is not behavioral strategy
    - LL with probability 0.4 and RR with probability 0.6 (why this is not behavioral?)

Example: Sequential Matching Pennies

- Consider following extensive form version of matching pennies



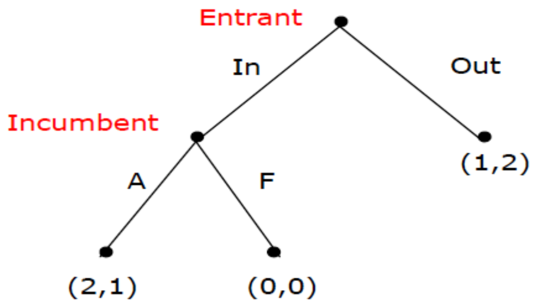
- How many strategies does agent 2 have?
  - $s_2 \in S_2 = \{HH, HT, TH, TT\}$
- Extensive form games can be represented as normal form games

Agent 1 \ Agent 2	HH	HT	TT	TH
Heads	(-1, 1)	(-1, 1)	(1, -1)	(1, -1)
Tails	(1, -1)	(-1, 1)	(-1, 1)	(1, -1)

- What will happen in this game?

Example: Entry Deterrence Game

- Consider following extensive form game



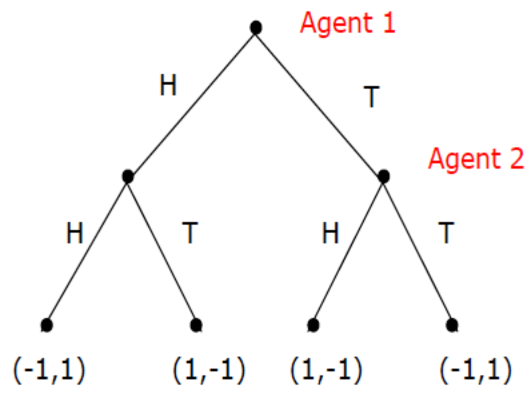
- What is equivalent strategic form representation?

Entrant \ Incumbent	A	F
In	(2, 1)	(0, 0)
Out	(1, 2)	(1, 2)

  - Two pure Nash equilibrium: (In, A) and (Out, F)
- Are Nash equilibria of this game reasonable in reality?
  - (Out, F) is sustained by noncredible threat of Entrant

Subgames

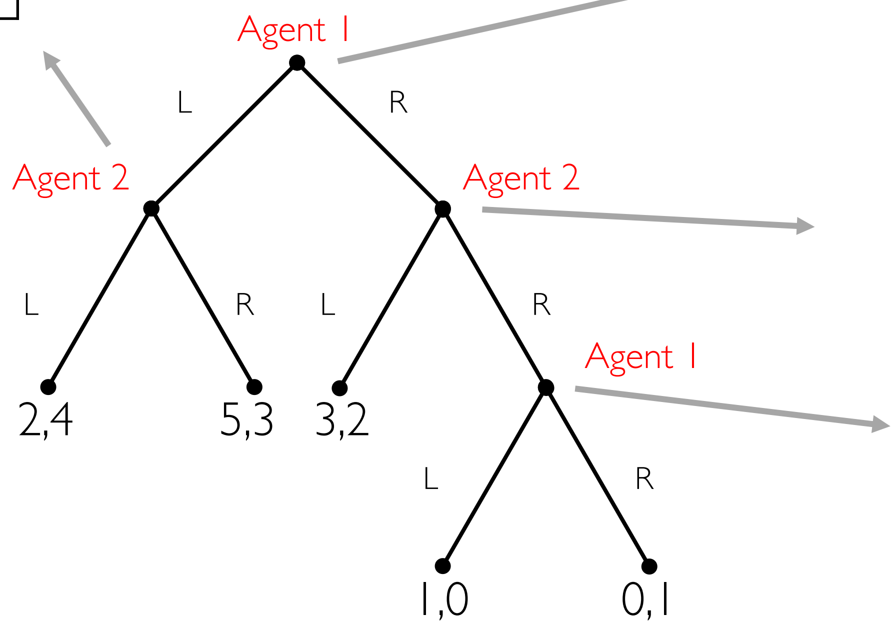
- Suppose that V_G represents set of all nodes in G's game tree
- Subgame G' of G consists of one choice node and all its successors
- Restriction of strategy s to subgame G' is denoted by $s_{G'}$
- Subgame G' can be analyzed as its own game



- Example: sequential matching pennies
  - How many subgame does this game have?
  - Given that game itself is also considered as subgame, there are three subgames

Matrix Representation of Subgames

Agent 1 \ Agent 2	L*	R*
**	2, 4	5, 3



Agent 1 \ Agent 2	LL	LR	RL	RR
LL	2, 4	2, 4	5, 3	5, 3
LR	2, 4	2, 4	5, 3	5, 3
RL	3, 2	1, 0	3, 2	1, 0
RR	3, 2	0, 1	3, 2	0, 1

Agent 1 \ Agent 2	*L	*R
*L	3, 2	1, 0
*R	3, 2	0, 1

Agent 1 \ Agent 2	**
*L	1, 0
*R	0, 1

Subgame Perfect Equilibrium (SPE)

- Profile s^* is SPE of game G if for any subgame G' of G, $s^*_{G'}$ is NE of G'
- Loosely speaking, subgame perfection will remove noncredible threats
  - Noncredible threads are not NE in their subgames
- How to find SPE?
  - One could find all of NE, then eliminate those that are not subgame perfect
  - But there are more economical ways of doing it

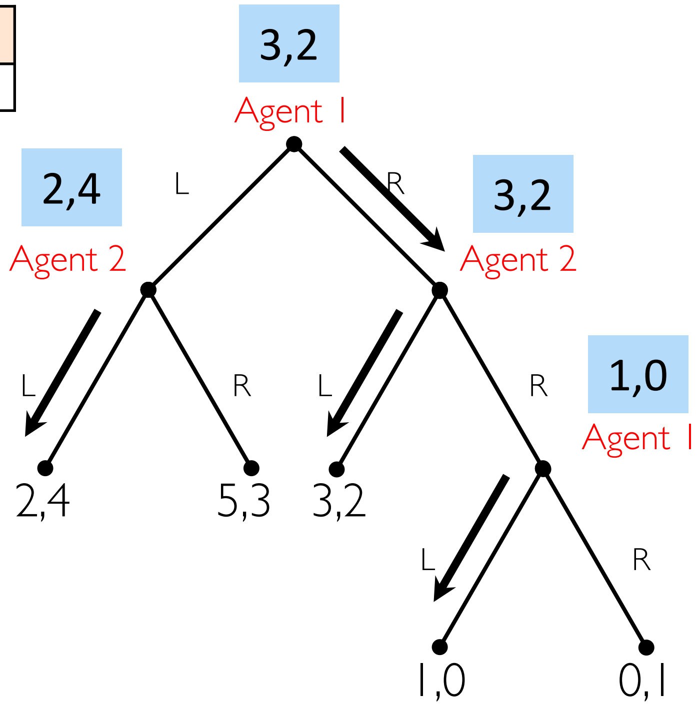
Backward Induction for Finite Games

- (1) Start from "last" subgames (choice nodes with all terminal children)
- (2) Find Nash equilibria of those subgames
- (3) Turn those choice nodes to terminal nodes using NE utilities
- (4) Go to (1) until no choice node remains

- [Theorem] Backward induction gives entire set of SPE

SPE of Extensive Form Game and NE of Subgames

Agent 2 / Agent 1	L*	R*
**	2, 4	5, 3



Agent 2 / Agent 1	LL	LR	RL	RR
LL	2, 4	2, 4	5, 3	5, 3
LR	2, 4	2, 4	5, 3	5, 3
RL	3, 2	1, 0	3, 2	1, 0
RR	3, 2	0, 1	3, 2	0, 1

Agent 2 / Agent 1	*L	*R
*L	3, 2	1, 0
*R	3, 2	0, 1

Agent 2 / Agent 1	**
*L	1, 0
*R	0, 1

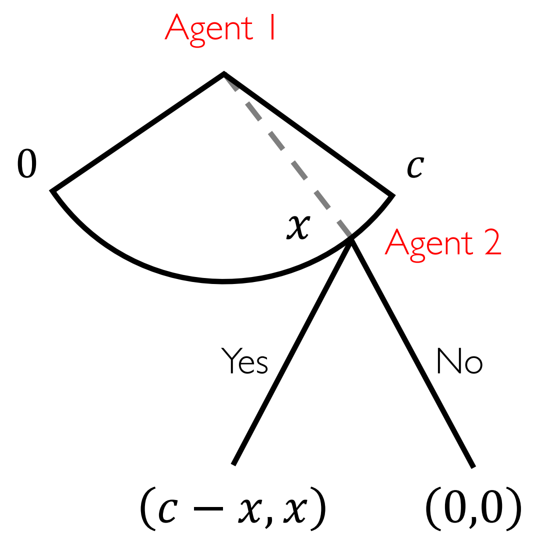
- (RR, LL) and (LR, LR) are not subgame perfect equilibria because (*R, **) is not an equilibrium
- (LL, LR) is not subgame perfect because (*L, *R) is not an equilibrium, *R is not a credible threat

Example: Stackleberg Model of Competition

- Consider variant of Cournot game where firm 1 first chooses q_1, then firm 2 chooses q_2 after observing q_1 (firm 1 is Stackleberg leader)
- Suppose that both firms have marginal cost c and inverse demand function is given by $P(Q) = \alpha - \beta Q$, where $Q = q_1 + q_2$, and $\alpha > c$
- Solve for SPE by backward induction starting firm 2's subgame
  - Firm 2 chooses $q_2 = \arg\max_{q \geq 0} \left(\alpha - \beta(q_1 + q) - c\right)q$
    - $q_2 = (\alpha - c - \beta q_1)/2\beta$
  - Firm 1 chooses $q_1 = \arg\max_{q \geq 0} \left(\alpha - \beta(q + (\alpha - c - \beta q)/2\beta) - c\right)q$
    - $q_1 = (\alpha - c)/2\beta$
    - $q_2 = (\alpha - c)/4\beta$

Example: Ultimatum Game

- Two agents want to split c dollars
  - I offers 2 some amount $x \leq c$
  - If 2 accepts, outcome is $(c - x, x)$
  - If 2 rejects, outcome is $(0, 0)$
- What is 2's best response if $x > 0$?
  - Yes
- What is 2's best response if $x = 0$?
  - Indifferent between Yes or No
- What are 2's optimal strategies?
  - (a) Yes for all $x \geq 0$
  - (b) Yes if $x > 0$, No if $x = 0$



SPE of Ultimatum Game

- What is 1's optimal strategy for each of 2's optimal strategies?
  - For (a), 1's optimal strategy is to offer $x = 0$
  - For (b),
    - If agent 1 offers $x = 0$, then her utility is 0
    - If she wants to offer any $x > 0$, then she must offer $\arg\max_{x>0} (c - x)$
      - This optimization does not have any optimal solution!
    - No offer of agent 1 is optimal!
- Unique SPE of ultimatum game is:

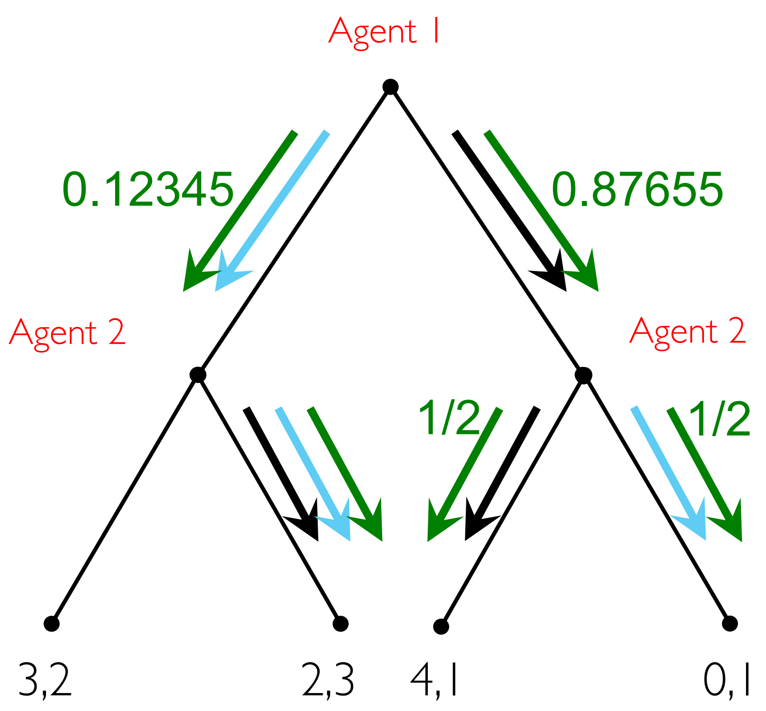
"Agent 1 offers 0, and agent 2 accepts all offers"

Modified Ultimatum Game

- If c is in multiples of cent, what are 2's optimal strategies?
  - (a) Yes for all $x \geq 0$
  - (b) Yes if $x > 0$, No if $x = 0$
- What are 1's optimal strategies for each of 2's?
  - For (a), offer $x = 0$
  - For (b), offer $x = 1$ cent
- What are SPE of modified ultimatum game?
  - Agent 1 offers 0, and agent 2 accepts all offers
  - Agent 1 offers 1 cent, and agent 2 accept all offers except 0
- Show that for every $\bar{x} \in [0, c]$, there exists NE in which 1 offers $\bar{x}$
  - What is agent 2's optimal strategy?

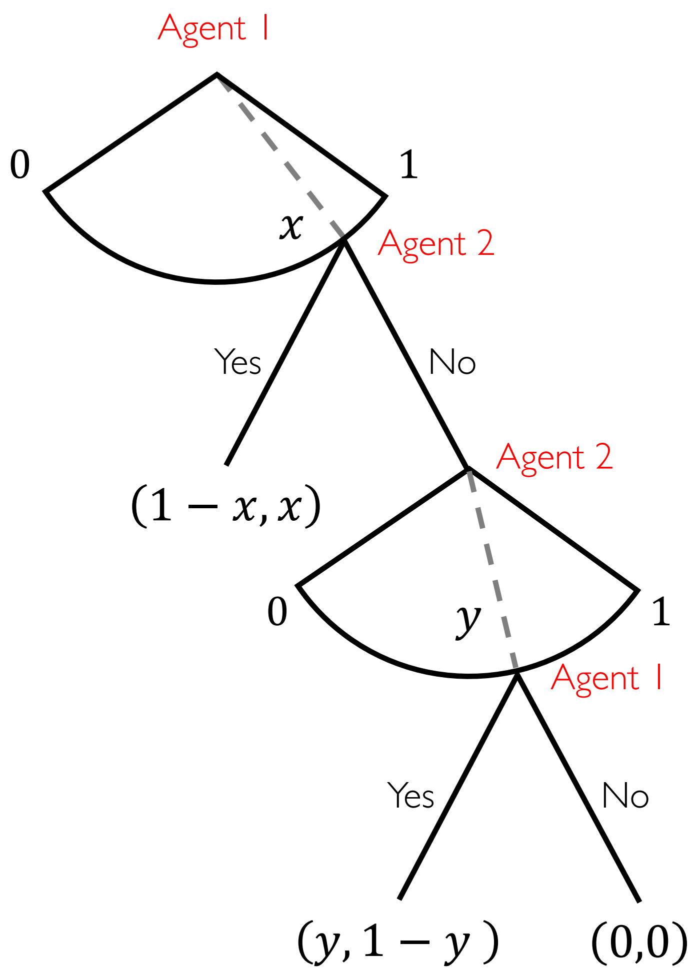
limitation of Backward Induction

- If there are ties, how they are broken affects what happens up in tree
  - There could be too many equilibria



Example: Bargaining Game

- Two agents want to split $c = 1$ dollar
- First, 1 makes her offer
- Then, 2 decides to accept or reject
- If 2 rejects, then 2 makes new offer
- Then, 1 decides to accept or reject
- Let $x = (x_1, x_2)$ with $x_1 + x_2 = 1$ denote allocations in 1st round
- Let $y = (y_1, y_2)$ with $y_1 + y_2 = 1$ denote allocations in 2nd round

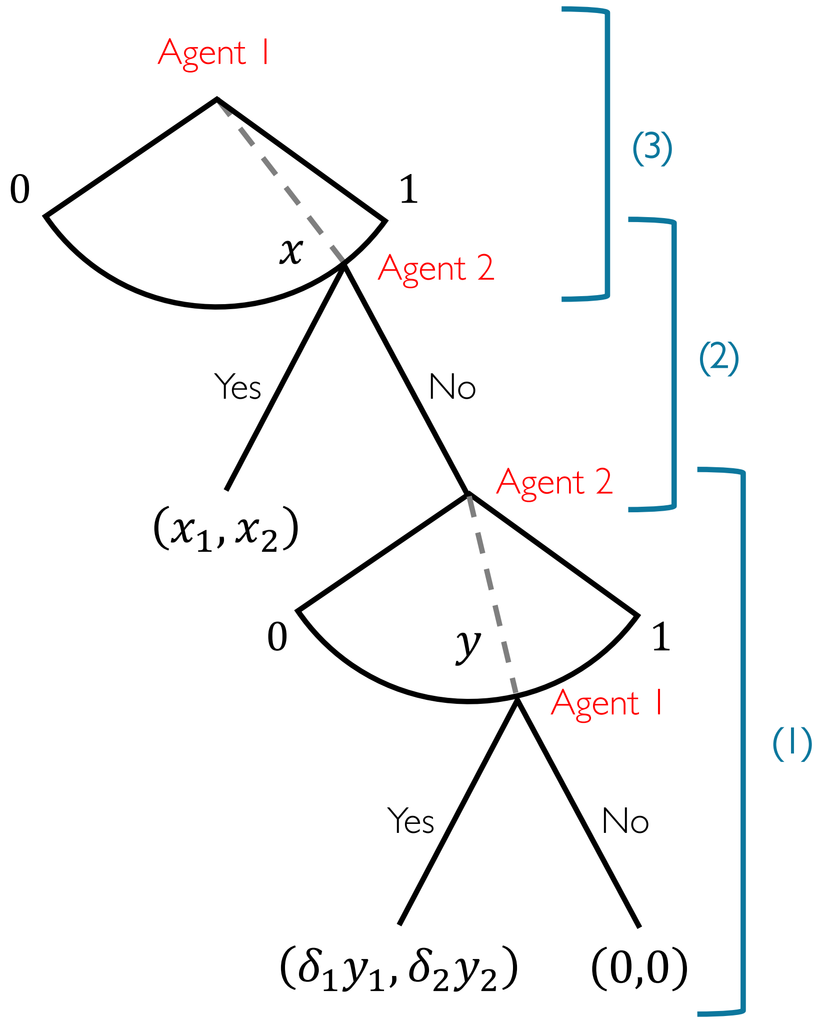


Backward Induction for Bargaining Game

- Second round is ultimatum game with unique SPE
  - Agent 2 offers $\mathbf{0}$, and agent 1 accepts all offers
- What is 2's optimal strategy in her round 1's subgame?
  - (a) If $x_2 \leq 1$, reject
  - (b) If $x_2 = 1$, accept, and reject otherwise
- What are 1's optimal strategies in round 1 for each of 2's?
  - For both (a) and (b), agent 1 is indifferent between all strategies
  - Agent 1's weakly dominant strategy is to offer $x_2 = 1$
- How many SPE does this game have?
  - Infinitely many! In all SPE, agent 2 gets everything
  - Last mover's advantage: In every SPE, agent who makes offer in last round obtains everything

Example: Discounted Bargaining Game

- Suppose utilities are discounted every round by discount factor, $0 < \delta_i < 1$
- What is unique SPE of (1)?
  - 2 offers $y_1 = 0$ and 1 accepts all offers
- What are optimal strategies in (2)?
  - (a) Yes if $x_2 \geq \delta_2$, No otherwise
  - (b) Yes if $x_2 > \delta_2$, No otherwise
- What are optimal strategies in (3)?
  - For (a), offer $x_2 = \delta_2$
  - For (b), there is no optimal strategy



Unique SPE of Discounted Bargaining Game

- What are SPE strategies?
  - Agent 1's proposes $(1-\delta_2, \delta_2)$
  - Agent 2 only accepts proposals with $x_2 \geq \delta_2$
  - Agent 2 proposes $(0,1)$ after any history in which 1's proposal is rejected
  - Agent 1 accepts all proposals of Agent 2
- What is SPE outcome of game?
  - Agent 1 proposes $(1-\delta_2, \delta_2)$
  - Agent 2 accepts
  - Resulting utilities are $(1-\delta_2, \delta_2)$
- Desirability of earlier agreement yields positive utility for agent 1

Stahl's Bargaining Model (for Finite Horizon Games)

- 2 rounds: $(1-\delta_2)$
- 3 rounds: $(1-\delta_2)+\delta_1\delta_2$
- 4 rounds: $(1-\delta_2)(1+\delta_1\delta_2)$
- 5 rounds: $(1-\delta_2)(1+\delta_1\delta_2)+\delta_1\delta_2$
- 2k rounds: $(1-\delta_2)\left(\frac{1-(\delta_1\delta_2)^k}{1-\delta_1\delta_2}\right)$
- 2k+1 rounds: $(1-\delta_2)\left(\frac{1-(\delta_1\delta_2)^k}{1-\delta_1\delta_2}\right)+(\delta_1\delta_2)^k$
- Taking limit as $k \to \infty$, we see that agent 1 gets $x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$ at SPE

Rubinstein's Infinite Horizon Bargaining Model

- Suppose agent can alternate offers forever
- There are two types of outcome to consider
  - At round t, one agent accepts her offer $(x^1, \text{No}, x^2, \text{No}, \dots, x^t, \text{Yes})$
  - Every offer gets rejected: $\left(x^1, \text{No}, x^2, \text{No}, \dots, x^k, \text{No}, \dots\right)$
- This is not finite horizon game, backward induction cannot be used
- We need different method to verify any SPE

One-Shot Deviation Principle

- One-shot deviation from strategy s means deviating from s in single stage and conforming to it thereafter

- Strategy profile s^* is SPE if and only if there exists no profitable one-shot deviation for each subgame and every agent

- This follows from principle of optimality of dynamic programming

SPE for Rubinstein's Model

- Recall that in Stahl's model, for $k \to \infty$, $x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$
- Is following strategy profile s^* SPE?
  - Agent 1 proposes x^* and accepts y if and only if $y \geq y_1^*$
  - Agent 2 proposes y^* and accepts x if and only if $x \geq x_2^*$
  - $x^* = (x_1^*, x_2^*)$, $x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$, $\quad x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$
  - $y^* = (y_1^*, y_2^*)$, $y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$, $\quad y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2}$

One-Shot Deviation Principle for Rubinstein's Model

- First note that this game has two types of subgames
  - (1) first move is offer
  - (2) first move is response to offer
- For (1), suppose offer is made by agent 1
  - If agent 1 adopts s^*, agent 2 accepts, agent 1 gets x_1^*
  - If agent 1 offers $> x_2^*$, agent 2 accepts, and agent 1 gets $x_1 < x_1^*$
  - If agent 1 offers $< x_2^*$, agent 2 rejects and offers y_1^*, agent 1 accepts and gets $\delta_1 y_1^* < x_1^*$
- For (2), suppose agent 1 is responding to offer $y_1 \geq y_1^*$
  - If agent 1 adopts s^*, she accepts and gets y_1
  - If agent 1 rejects and offers x_2^* in next round, agent 2 accepts, agent 1 gets $\delta_1 x_1^* = y_1^* \leq y_1$
- For (2), suppose agent 1 is responding to offer $y_1 < y_1^*$
  - If agent 1 adopts s^*, she rejects and offers x_2^* in next round, agent 2 accepts, agent 1 gets $\delta_1 x_1^* = y_1^* > y_1$
  - If agent 1 accepts, she gets $y_1 < y_1^*$
- Hence s^* is SPE (in fact unique SPE, check GT, Section 4.4.2 to verify)

Rubinstein’s Model for Symmetric Agents

- Suppose that $\delta_1 = \delta_2$
  - If agent 1 moves first, division is $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$
  - If agent 2 moves first, division is $\left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$

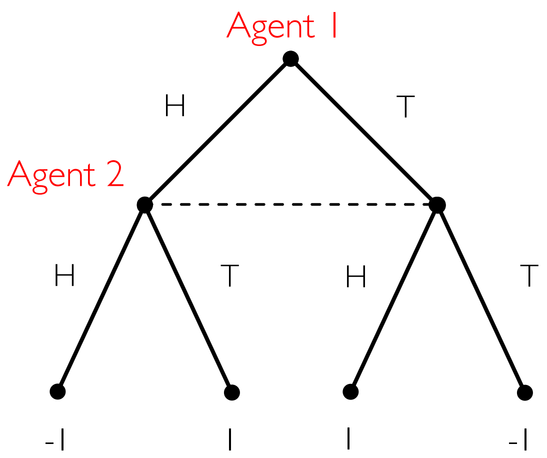
- First mover’s advantage is related to impatience of agents
  - If $\delta \to 1$, FMA disappears and outcome tends to $\left(\frac{1}{2}, \frac{1}{2}\right)$
  - If $\delta \to 0$, FMA dominates and outcome tends to $(1,0)$

Imperfect Information Extensive Form Games

- In perfect information games, agents know choice nodes they are in
  - Agents know all prior actions
  - Recall that in such games choice nodes are equal to histories that led to them
- Agents may have partial or no knowledge of actions taken by others
- Agents may also have imperfect recall of actions taken by themselves

Example: Imperfect Information Sequential Matching Pennies

- Agent 1 takes action
- Agent 2 does not see agent 1's action
- Agent 2 takes action, and outcome is revealed



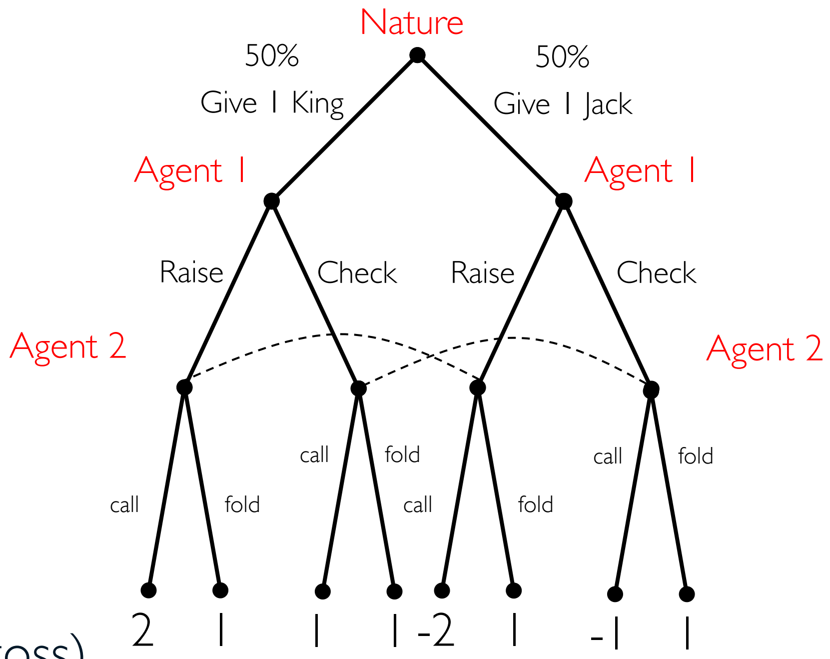
- Information set is collection of choice nodes that cannot be distinguished by agents whose turn it is
- Set of agents and their actions at each choice node in information set has to be the same, otherwise, agents could distinguish between nodes

Finite Imperfect-Information Extensive Form Games

- Formally, each game is tuple $G = \langle \mathcal{I}, (\mathcal{A}_i)_{i\in\mathcal{I}}, \mathcal{H}, \mathcal{Z}, \alpha, (\beta_i)_{i\in\mathcal{I}}, \rho, (u_i)_{i\in\mathcal{I}}, I \rangle$
  - $\langle \mathcal{I}, (\mathcal{A}_i)_{i\in\mathcal{I}}, \mathcal{H}, \mathcal{Z}, \alpha, (\beta_i)_{i\in\mathcal{I}}, \rho, (u_i)_{i\in\mathcal{I}} \rangle$ is perfect information, extensive form game
  - $I = (I_1, \ldots, I_n)$, where $I_j = \left\{h_{j,1}, \ldots, h_{j,k_j}\right\}$, is partition of $\mathcal{H}$ such that if $h, h' \in I_j$, then $\alpha(h) = \alpha(h')$, and for all $i \in \alpha(h)$, $\beta_i(h) = \beta_i(h')$

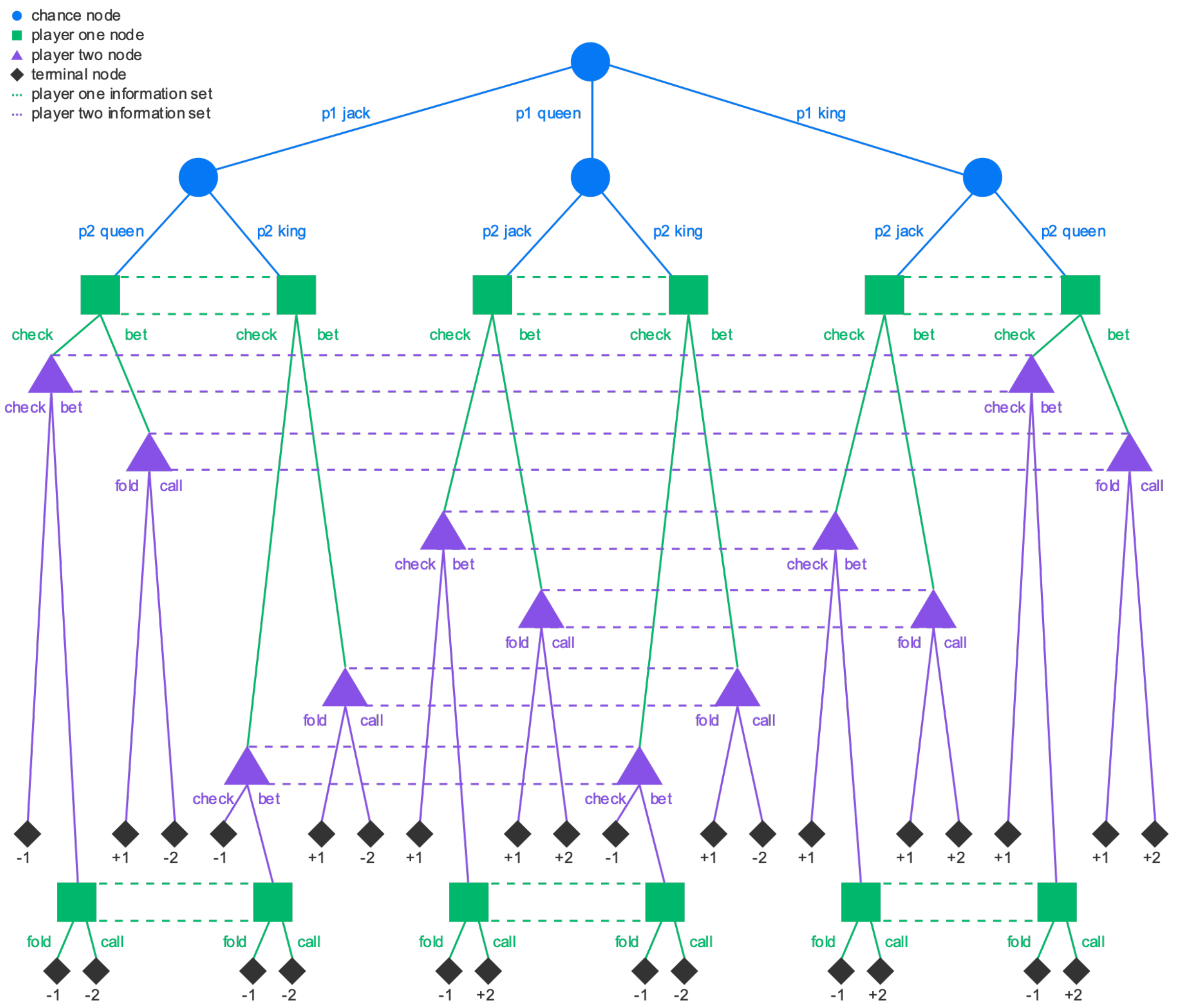
Example: Poker-Like Game

- What are agent 1's strategies?
  - $\{RR, RC, CR, CC\}$
- What are agent 2's strategies?
  - $\{CC, CF, FC, FF\}$
- How can we find NE of this game?
  - Model game as normal form zero-sum game
    - Each cell represents expected utilities (nature's coin toss)
  - Eliminated (weakly) dominated strategies
  - Solve for (mixed strategy) NE



	Agent 2 / Agent 1	CC	~~CF~~	FC	~~FF~~
		2/3		1/3	
1/3	RR	0, 0	~~0, 0~~	1, -1	~~1, -1~~
2/3	RC	0.5, -0.5	~~1.5, -1.5~~	0, 0	~~1, -1~~
	~~CR~~	~~-0.5, 0.5~~	~~-0.5, 0.5~~	~~1, -1~~	~~1, -1~~
	~~CC~~	~~0, 0~~	~~1, -1~~	~~0, 0~~	~~1, -1~~

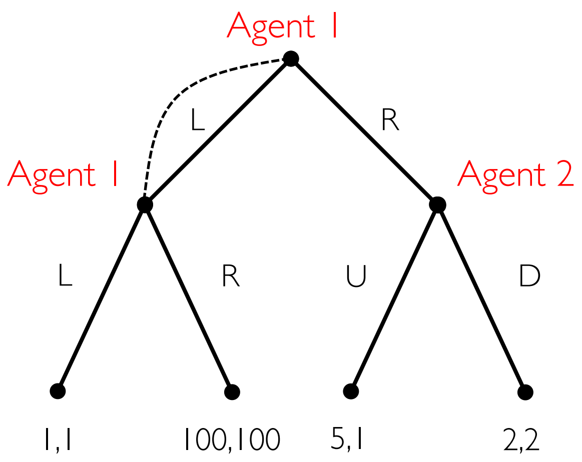
Example: Kune Poker



https://justinsermeno.com/posts/cfr/

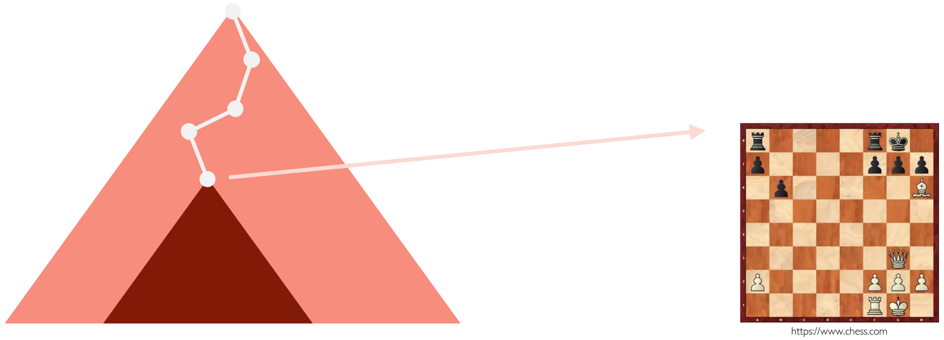
Imperfect Recall, Mixed vs Behavioral Strategies

- Consider mixed strategies
  - What is NE of this game?
    - (R,D) with outcome utilities (2,2)
- Consider behavioral strategies
  - What is 1's expected utility if she does $(p, 1-p)$
    - $p^2 + 100p(1-p) + 2(1-p)$
  - What is 1's best response?
    - $p = \frac{98}{198}$
  - What is NE of this game?
    - $\left(\left(\frac{98}{198}, \frac{100}{198}\right), (0,1)\right)$



Solving Extensive Form Games: Perfect vs Imperfect Information

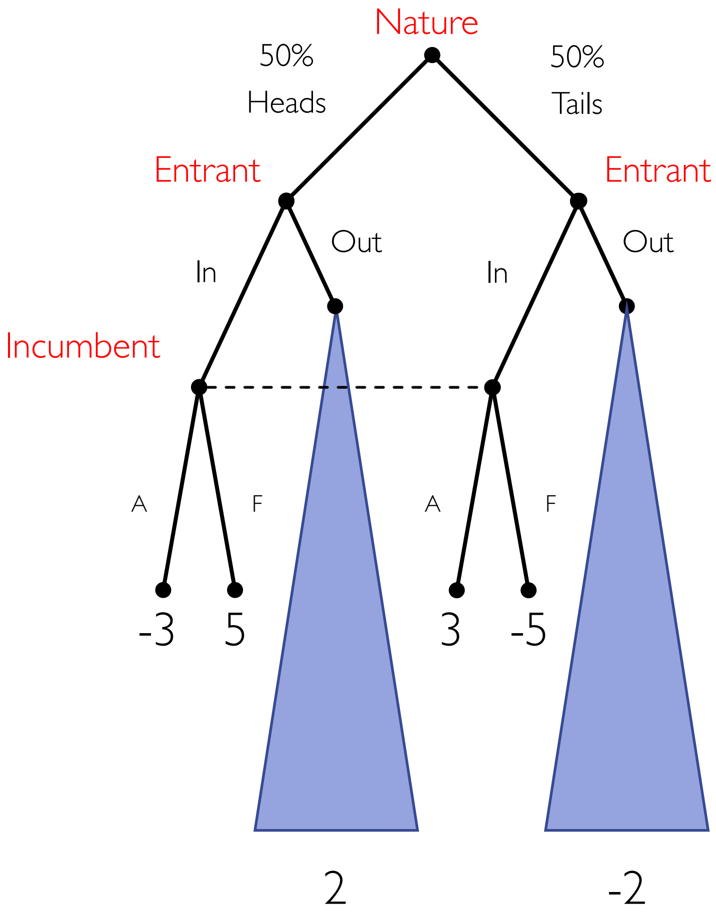
- In perfect information games, optimal strategy for each subgame can be determined by that subgame alone (how backward induction works!)
  - We can forget how we got here
  - We can ignore rest of game



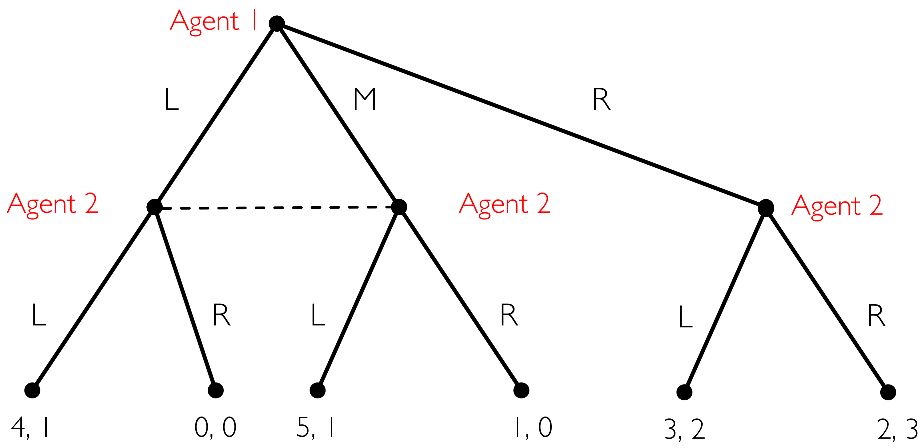
- In imperfect information games, this is not necessarily true
  - We cannot forget about path to current node
  - We cannot ignore other subgames

Example

- Is always accommodating good strategy?
  - No, leads to utility of -2.5 for incumbent
- Is always fighting good strategy?
  - No, leads to utility of -1.5 for incumbent
- What should incumbent do?
  - A with 3/8 probability and F with 5/8
- What if we swap 2 and -2?
  - A with 7/8 probability and F with 1/8



Subgame Perfection and Imperfect Information



- There are two subgames: game itself and subgame after agent 1 plays R
  - (R, RR) is NE and SPE
- But, why should 2 play R after 1 plays L/M?
  - This is noncredible threat
- There are more sophisticated equilibrium refinements that rule this out

Questions?

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