ECE700.07: Game Theory with Engineering Applications

Lecture 7: Games with Incomplete Information

Seyed Majid Zahedi

UNIVERSITY OF WATERLOO

Outline

- Bayesian games
- Bayes-Nash equilibrium
- Auctions

- Readings:
  - MAS Sec. 6.3, GT Sec. 6.1 - 6.5

Bayesian Games: Games of Incomplete Information

- So far, we assumed all agents know what game they are playing
  - Number of agents
  - Actions available to each agent
  - Utilities associated with each outcome
- In extensive form games, actions may not be common knowledge, but game itself is
- *Bayesian games* allow us to represent agents' uncertainties about game being played
  - Uncertainty is represented as commonly known probability distribution over possible games
- We make following assumptions
  - All games have same number of agents and same strategy space for each agents
  - Possible games only differ in agents' utilities for each outcome
  - Beliefs are posteriors, obtained by conditioning common prior on private signals

Example: Bayesian Entry Deterrence Game

- Incumbent decides whether to build new plant, entrant decides whether to enter
- Incumbent knows her cost, entrant is uncertain if incumbent's building cost is 4 or 1
- Game takes one of following two forms

Incumbent \ Entrant	Enter	Stay Out
Build	(0, -1)	(2, 0)
Don't Build	(2, 1)	(3, 0)

High Building Cost

Incumbent \ Entrant	Enter	Stay Out
Build	(3, -1)	(5, 0)
Don't Build	(2, 1)	(3, 0)

Low Building Cost

- Suppose entrant assigns prior probability of p to incumbent's cost being high
- Incumbent's dominant strategy is "build" if cost is low and "don't build" otherwise
- Entrant's utility is $2p - 1$ for "enter" and and 0 for "stay out"
- Entrant enters if $p > 1/2$

Example: Bayesian Entry Deterrence Game (cont.)

- Now suppose entrant is uncertain if incumbent's building cost is 4 or 2.5

Incumbent \ Entrant	Enter	Stay Out
Build	(0, -1)	(2, 0)
Don't Build	(2, 1)	(3, 0)

High Building Cost

Incumbent \ Entrant	Enter	Stay Out
Build	(1.5, -1)	(3.5, 0)
Don't Build	(2, 1)	(3, 0)

Low Building Cost

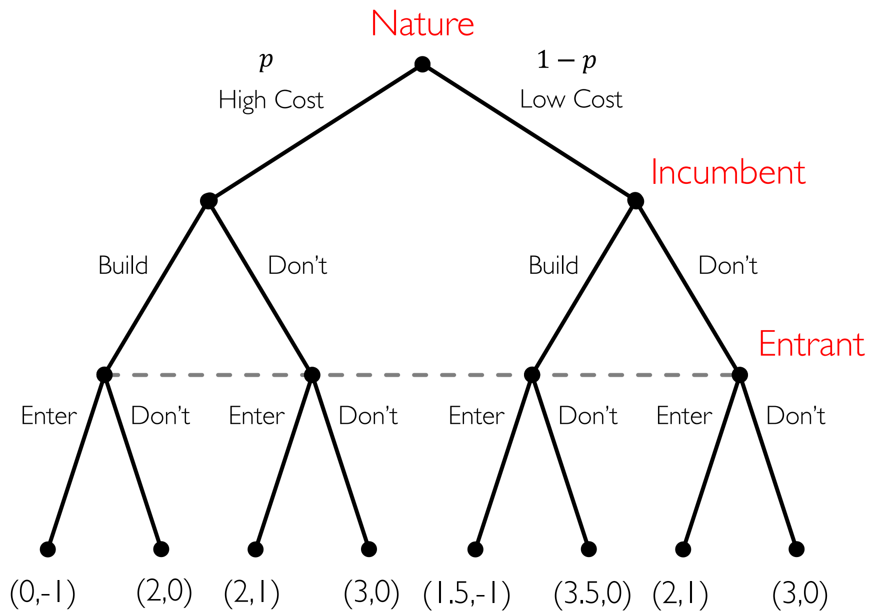
- "Don't build" is still dominant strategy for incumbent if cost is high
- Incumbent's strategy if cost is low depends on her prediction of entrant's strategy
- If y is incumbent's prediction of entrant playing "enter", then "build" is better if

$$1.5y + 3.5(1-y) > 2y + 3(1-y) \Rightarrow y < 1/2$$

- Incumbent must predict entrant's strategy
- Entrant cannot infer incumbent's strategy only from her knowledge of utilities

Example: Bayesian Entry Deterrence Game (cont.)

- We can model game as extensive form game
  - Nature chooses incumbent's type
  - Agents have same prior belief about nature's move
- Suppose that
  - Incumbent chooses build with probability x if cost is low
  - Entrant chooses enter with probability y
- What is incumbent's best response to y if cost is low?
  - $x = 1$ if $y < 1/2$ and $x = 0$ if $y > 1/2$
  - $x \in [0,1]$ if $y = 1/2$
- What is entrant's best response to x?
  - $y = 1$ if $x < 1/2(1-p)$ and $y = 0$ if $x > 1/2(1-p)$
  - $y \in [0,1]$ if $x = 1/2(1-p)$
- Search for Bayes-Nash equilibrium boils down to finding (x, y) that are optimal for both
  - $(0,1)$ for any p or $(1,0)$ if and only if $p \leq 1/2$ or $(1/2(1-p), 1/2)$



Bayesian Games Model

- Bayesian game is tuple $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (\Theta_i)_{i\in\mathcal{I}}, p, (u_i)_{i\in\mathcal{I}} \rangle$
  - $\mathcal{I}$ is finite set of agents
  - S_i is set of actions available to agent i
  - Θ_i is type space of agent i
  - $p: \Theta \longmapsto [0,1]$ is common prior over types
  - $u_i: S \times \Theta \longmapsto \mathbb{R}$ is utility function for agent i
- Agent i's mixed strategy $\sigma_i: \Theta_i \rightarrow \Sigma_i$ is contingency plan for all $\theta_i \in \Theta_i$
  - $\sigma_i(\theta_i)$ specifies i's mixed strategy when her type is θ_i
  - $\sigma_i(s_i|\theta_i)$ specifies probability of agent i taking action s_i when her type is θ_i

Expected Utilities

- Ex-post expected utility

$$EU_i(\sigma, \theta) = \sum_{s \in S} \left(\prod_{j \in \mathcal{I}} \sigma_j(s_j | \theta_j) \right) u_i(s, \theta)$$

- Ex-interim expected utility

$$EU_i(\sigma, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) EU_i(\sigma, (\theta_i, \theta_{-i}))$$

- Ex-ante expected utility

$$EU_i(\sigma) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(\sigma, \theta_i) = \sum_{\theta \in \Theta} p(\theta) EU_i(\sigma, \theta)$$

Example

- Consider following game which consists of four 2×2 games
  - Matching Pennies, Prisoner's Dilemma, Coordination and Battle of the Sexes

θ_{21} / θ_{11}: MP

2,0	0,2
0,2	2,0

$p = 0.3$

θ_{22} / θ_{11}: PD

2,2	0,3
3,0	1,1

$p = 0.1$

θ_{21} / θ_{12}: Coord

2,2	0,0
0,0	1,1

$p = 0.2$

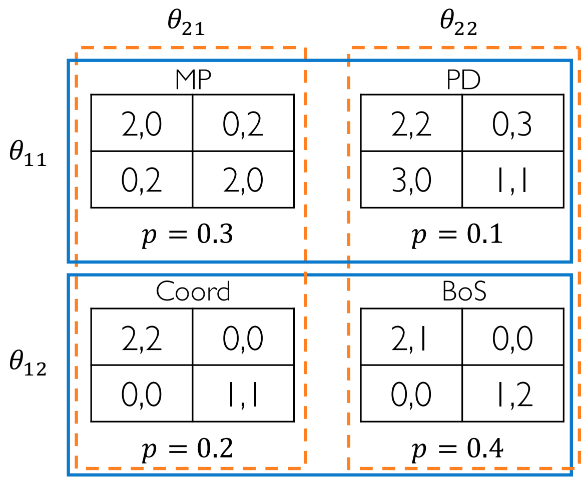
θ_{22} / θ_{12}: BoS

2,1	0,0
0,0	1,2

$p = 0.4$

Example (cont.)

- What is $EU_2(UD, LR)$?



$$EU_2(UD, LR) = \sum_{\theta \in \Theta} p(\theta) EU_2(UD, LR, \theta)$$

$$\begin{aligned} = \; & p(\theta_{11}, \theta_{2,1}) u_2(U, L, \theta_{11}, \theta_{2,1}) + p(\theta_{11}, \theta_{2,2}) u_2(U, R, \theta_{11}, \theta_{2,2}) + \\ & p(\theta_{12}, \theta_{2,1}) u_2(D, L, \theta_{12}, \theta_{2,1}) + p(\theta_{12}, \theta_{2,2}) u_2(D, R, \theta_{12}, \theta_{2,2}) + \end{aligned}$$

$$= \; 0.3 \times 0 + 0.1 \times 3 + 0.2 \times 0 + 0.4 \times 2 = 1.1$$

Strategies

- Agent i's best response correspondence to mixed strategy σ_{-i} is

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i^{|\Theta_i|}} EU_i(\sigma_i, \sigma_{-i})$$

$$= \arg\max_{\sigma_i \in \Sigma_i^{|\Theta_i|}} \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(\sigma_i, \sigma_{-i}, \theta_i)$$

  - $EU_i(\sigma_i, \sigma_{-i}, \theta_i)$ is independent of $\sigma_i(\theta_i')$ for all $\theta_i' \neq \theta_i$
  - Maximizing $EU_i(\sigma_i, \sigma_{-i})$ is equal to maximizing $EU_i(\sigma_i, \sigma_{-i}, \theta_i)$ for all $\theta_i \in \Theta_i$

Bayes-Nash Equilibrium

- Bayes-Nash equilibrium (BNE) is mixed strategy profile $\boldsymbol{\sigma}^*$, such that

$$\sigma_i^* \in BR_i(\sigma_{-i}^*), \quad \forall i$$

- BNE of Bayesian game are NE of its induced normal form game
- [Theorem] Any finite Bayesian game has mixed strategy BNE

Ex-Post Equilibrium

- Mixed-strategy profile σ^* is ex-post equilibrium if

$$\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i^{|\Theta_i|}} EU_i(\sigma_i, \sigma_{-i}^*, \theta) \quad \forall i, \theta \in \Theta$$

- Ex-post equilibrium is similar to dominant strategy equilibrium
  - Agents are not assumed to know $\boldsymbol{\theta}$
  - Even if they knew $\boldsymbol{\theta}$, agents would never want to deviate
  - Ex-post equilibrium is not guaranteed to exist

Example: Incomplete Information Cournot

- Two firms decide on their production level $q_i \in [0, \infty)$
- Price is given by $P(q)$ where $q = q_1 + q_2$
- Firm 1 has marginal cost equal to c which is common knowledge
- Firm 2's marginal cost is private information
  - c_L with probability x and c_H with probability $(1 - x)$, where $c_L < c_H$
- Utility of agents are ($t \in \{L, H\}$ type of firm 2)
  - $u_1\big((q_1, q_2), t\big) = q_1 P(q_1 + q_2) - c$
  - $u_2\big((q_1, q_2), t\big) = q_2 P(q_1 + q_2) - c_t$

Example: Incomplete Information Cournot (cont.)

- What is firm 1's best response to (q_L, q_H)?

$$B_1(q_L, q_H) = \arg\max_{q \geq 0} \Big(\big(xP(q + q_L) + (1 - x)P(q + q_H) - c\big)q \Big)$$

$$B_2^L(q_1) = \arg\max_{q \geq 0} \Big(\big(P(q_1 + q) - c_L\big)q \Big)$$

$$B_2^H(q_1) = \arg\max_{q \geq 0} \Big(\big(P(q_1 + q) - c_H\big)q \Big)$$

- BNE of this game is vector (q_1^*, q_L^*, q_H^*) such that

$$q_1^* \in B_1(q_L^*, q_H^*), q_L^* \in B_2^L(q_1^*), q_H^* \in B_2^H(q_1^*)$$

Auctions

- Major application of Bayesian games is in auctions
- Auctions are commonly used to sell (allocate) items to bidders
- Auctioneer often would like to maximize her revenue
- Bidders' valuations are usually unknown to others and auctioneer
- Allocating items to bidders with highest valuations is often desirable
- Extracting private valuations could be challenging
  - *E.g.*, giving painting for free to bidder with highest valuation would create incentive for all bidders to overstate their valuations

Different Auctions and Terminologies

- English auction: ascending sequential bids
- First price auction: bidders bid simultaneously, highest bid wins, winner pays her bid
- Second price action: similar to first price, except that winner pays second highest bid
- Dutch auction: descending sequential prices; price is reduced until one stops auction
- Private valuations: valuation of each bidder is independent of others' valuations
- Common valuations: bidders' valuations are imperfectly correlated to common value

Modeling First and Second Price Auctions

- Suppose that there are N bidders and single object for sale
- Bidder i has value v_i for object and bids b_i
- Utility of bidder i is $v_i - p_i$, where p_i is bidder i's payment
- Suppose v's are drawn *i.i.d.* from $[0, \bar{v}]$ with commonly known CDF F
- Bidders only know their own realized value (type)
- Bidders are risk neutral, maximizing their expected utility
- Pure strategy for bidder i is map $b_i: [0, \bar{v}] \rightarrow \mathbb{R}_+$
- We focus on symmetric strategies

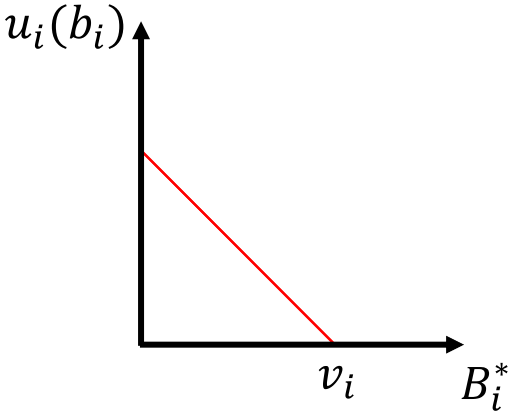
Second Price Auctions

- Agent i submit her bid, b_i, simultaneously with other agents
- Agent with highest bid wins, and pays second highest bid
- Agent i's profit is $v_i - b_j$ if she wins, and 0 otherwise

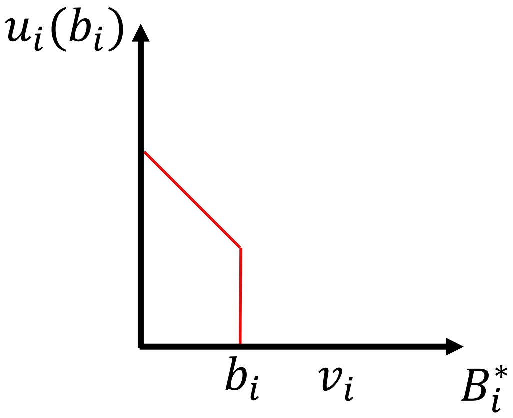
- [Proposition] Truthful bidding (i.e., $b_i = v_i$) is BNE in second price auction
- [Proof] We need to answer following questions
  - If other bidders bids truthfully, does winner want to change her bid?
  - If other bidders bids truthfully, does looser want to change her bid?

Truthful Bidding

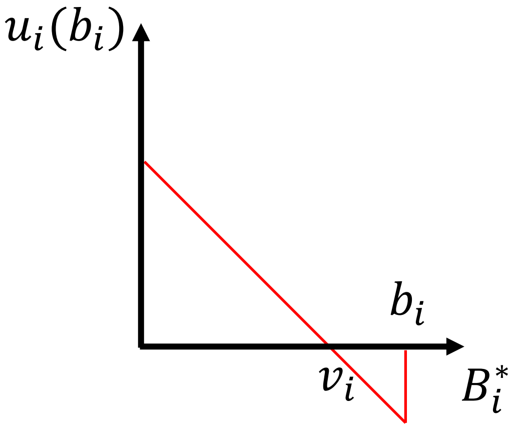
- Truthful equilibrium is (weak) ex-post equilibrium
  - Truthful bidding weakly dominates other strategies even if all values are known
- [Picture proof]
  - Suppose $B_i^* = \max_{j \neq i} b_j$ represents maximum bids excluding i's bid



$b_i = v_i$



$b_i < v_i$



$b_i > v_i$

Expected Payment in Second Price Auctions

- Define random variable y_i to be $\max_{j \neq i} v_j$
  - CDF of y_i is $G_{y_i}(v) = F(v)^{N-1}$
  - PDF of y_i is $g_{y_i}(v) = (N-1)f(v)F(v)^{N-2}$
- Expected payment of bidder i with value v_i is given by

$$\begin{aligned} p(v_i) &= \Pr(v_i \text{ wins}) \times \mathbb{E}[y_i | y_i \leq v_i] \\ &= \Pr(y_i \leq v_i) \times \mathbb{E}[y_i | y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy \\ &= \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$

First Price Auctions

- Bidder i submits bid b_i
- Utility of agent i is $v_i - b_i$ if $b_i > max_{j \neq i} b_j$ and zero otherwise
- We focus on symmetric (increasing and differentiable) equilibrium strategies β
- Note that bidder with value 0 always bids 0, i.e., $\beta(0) = 0$
- Bidder i wins whenever $max_{j \neq i} \beta(v_j) < b_i$
- Since β is increasing, we have $max_{j \neq i} \beta(v_j) = \beta(max_{j \neq i} v_j) = \beta(y_i)$
- This implies that bidder i wins whenever $y_i < \beta^{-1}(b_i)$

First Price Auctions (cont.)

- Optimal bid of bidder i is $b_i = \arg\max_{b \geq 0} G_{y_i}(\beta^{-1}(b))(v_i - b)$
- First-order (necessary) optimality conditions imply

$$\frac{g_{y_1}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}(v_i - b_i) - G_{y_i}(\beta'(b_i)) = 0$$

  - Note that derivative of $\beta^{-1}(b)$ is $1/\beta'\left(\beta^{-1}(b)\right)$
- In symmetric equilibrium, $b_i = \beta(v_i)$, therefore we have

$$v_i g_{y_i}(v_i) = \beta'(v_i) G_{y_i}(v_i) + \beta(v_i) g_{y_i}(v_i) = \frac{d}{dv}\left(\beta(v_i) G_{y_i}(v_i)\right)$$

- With boundary condition $\beta(0) = 0$, we have

$$\beta(v_i) = G_{y_i}^{-1}(v_i) \int_0^{v_i} y g_{y_i}(y) dy = \mathbb{E}[y_i | y_i \leq v_i]$$

Expected Payment in First Price Auctions

- Expected payment of bidder i with value v_i is

$$\begin{aligned} p(v_i) &= \Pr(v_i \text{ wins}) \times \beta(v_i) \\ &= \Pr(y_i \leq v_i) \times \mathbb{E}[y_i | y_i \leq v_i] \\ &= G_{y_i}(v_i) \times G_{y_i}(v_i)^{-1} \int_0^{v_i} y g_{y_i}(y) dy \\ &= \int_0^{v_i} y g_{y_i}(y) dy \end{aligned}$$

- This establishes somewhat surprising results that both first and second price auction formats yield same expected revenue to auctioneer

Revenue Equivalence

- In standard auctions item is sold to bidder with highest submitted bid
- Suppose that values are *i.i.d* and all bidders are risk neutral

- [Theorem] Any symmetric and increasing equilibria of any standard auction (such that expected payment of bidder with value zero is zero) yields same expected revenue to auctioneer

Common Value Auctions (Dependent Signals)

- In common value auctions, value of item for sale is same for all bidders
- Suppose that there are two bidders bidding to lease oil field
- Oil field could be worth \$0 (25%), \$25M (50%), or \$50M (25%)
- Bidders hires their own consultant to evaluate value of oil field
  - Bidder 1 gets private information (signal) s_1
  - Bidder 2 gets private information (signal) s_2
- Suppose that signals are correlated with value of oil field as follows
  - If field is worth \$0, then $s_1 = s_2 = L$
  - If field is worth \$25M, then $s_1 = H, s_2 = L$ or $s_1 = L, s_2 = H$ (both equally likely)
  - If field is worth \$50M, then $s_1 = s_2 = H$
- Given their private signals, how should bidders bid?

Oil Field Example: Expected Value

- What is expected value of oil field if one receives L signal?
  - Given L signal, oil field is worth either \$0 or \$25

$$\Pr(\$25\text{M}|L) = \frac{\Pr(\$25\text{M}) \times \Pr(L|\$25\text{M})}{\Pr(\$25\text{M}) \times \Pr(L|\$25\text{M}) + \Pr(\$0) \times \Pr(L|\$0)} = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.25 \times 1} = 0.5$$

$$\Pr(\$0|L) = \frac{\Pr(\$0) \times \Pr(L|\$0)}{\Pr(\$25\text{M}) \times \Pr(L|\$25\text{M}) + \Pr(\$0) \times \Pr(L|\$0)} = \frac{0.25 \times 1}{0.5 \times 0.5 + 0.25 \times 1} = 0.5$$

$$\mathbb{E}[\text{oil field's value}|L] = \$25\text{M} \times \Pr(\$25\text{M}|L) + \$0 \times \Pr(\$0|L) = \$12.5\text{M}$$

$$\mathbb{E}[\text{oil field's value}|H] = \$50\text{M}\times\Pr(\$50\text{M}|H)+\$25\text{M}\times\Pr(\$25\text{M}|H) = \$37.5\text{M}$$

Oil Field Example: Second Price Auction and Truthful Bidding

- What is expected utility of bidding \$12.5M upon receiving L?
  - With probability 0.5, true value is \$0
    - Other bidder bids \$12.5M
    - Each bidder wins with probability 0.5 and gets -\$12.5M
  - With probability 0.5, true value is \$25M
    - Other bidder bids \$37.7M
    - Bidder with L loses and gets \$0
  - Expected utility = 0.5 × 0.5 × (-\$12.5M)
- Bidding \$0 leads to utility \$0 and is profitable deviation
- Truthful bidding is not BNE in second price auction with common values and dependent signals

Winner's Curse

- Winning means bidder received highest or most optimistic signal
- Condition on winning, value of item is lower than what signal says
- Ignoring this leads to paying, on average, more than true value of item
- To avoid this curse, bidders should assume their signal is optimistic
- In oil field example, we can show that following bidding strategy is BNE
  - Bid 0 upon receiving L
  - Bid $50M upon receiving H

Common Value Auctions (Independent Signals)

- Consider two bidders interested in buying oil field that has part A and B
- Each bidder values A and B but is more interested in one of them
- Bidders hires their own consultant to evaluate value of their part
  - Bidder 1 gets private signal s_1 about value of part A
  - Bidder 2 gets private signal s_2 about value of part B
- Suppose that both signals are uniformly distributed over $[0,1]$
- Suppose value of oil field to each bidder is as follows
  - $v_i = as_i + bs_{-i}$ with $a \geq b \geq 0$
  - Private values are special case where $a = 1$ and $b = 0$

Oil Field Example II: Second Price Auction and Truthful Bidding

- Similar to previous example, truthful bidding is not BNE
- Instead, we show that following symmetric bidding strategy is BNE

$$\beta(s_i) = (a+b)s_i$$

- If other bidder follows this, then probability that i wins by bidding b_i is

$$\Pr\big(\beta(s_{-i}) < b_i\big) = \Pr\big((a+b)s_{-i} < b_i\big) = b_i/(a+b)$$

  - Bidder i's payment if she wins is

$$\beta(s_{-i}) = (a+b)s_{-i}$$

Oil Field Example II: Second Price Auction and Truthful Bidding (cont.)

- Expected payment of bidder i is

$$\mathbb{E}\big[(a+b)s_{-i}|s_{-i} < b/(a+b)\big] = b/2$$

- Expected utility of bidding b_i with signal s_i is

$$EU(b_i, \beta, s_i) = \Pr[b_i \text{ wins}] \times \Big(as_i + b\mathbb{E}\big[(a+b)s_{-i}|b_i \text{ wins}\big] - b_i/2\Big)$$

$$= b_i/(a+b) \times (as_i + bb_i/2(a+b) - b_i/2)$$

- Maximizing this with respect to b_i (for given s_i) implies

$$\beta(s_i) = (a+b)s_i$$

Oil Field Example II: First Price Auction

- Analysis is similar to that of first price auctions with private values
- It can be shown that unique symmetric BNE is for each bidder to bid

$$\beta(s_i) = \frac{1}{2}(a+b)s_i$$

- It can be shown that expected revenue is equal to first price auction
- Revenue equivalence principle continues to hold for common values

Questions?

Acknowledgement

- This lecture is a slightly modified version of one prepared by
  - Asu Ozdaglar [MIT 6.254]