Outline

• Strategic form games
• Dominant strategy equilibrium
• Pure and mixed Nash equilibrium
• Iterative elimination of strictly dominated strategies
• Price of anarchy
• Correlated equilibrium

• Readings:
  • MAS Sec. 3.2 and 3.4, GT Sec. 1 and 2
Strategic Form Games

• Agents act simultaneously without knowledge of others’ actions

• Each game has to have
  • (1) Set of agents (2) Set of actions (3) Utilities

• Formally, strategic form game is triplet $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$
  • $\mathcal{I}$ is finite set of agents
  • $S_i$ is set of available actions for agent $i$ and $s_i \in S_i$ is action of agent $i$
  • $u_i: S \rightarrow \mathbb{R}$ is utility of agent $i$, where $S = \prod_i S_i$ is set of all action profiles
  • $s_{-i} = [s_j]_{j \neq i}$ is vector of actions for all agents except $i$
  • $S_{-i} = \prod_{j \neq i} S_j$ is set of all action profiles for all agents except $i$
  • $(s_i, s_{-i}) \in S$ is strategy profile, or outcome
Example: Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Stay Silent</th>
<th>Confess</th>
</tr>
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<tbody>
<tr>
<td>Stay Silent</td>
<td>(-1, -1)</td>
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- First number denotes utility of A1 and second number utility of A2
- Row $i$ and column $j$ cell contains $(x, y)$, where $x = u_1(i, j)$ and $y = u_2(i, j)$
Strategies

• Strategy is complete description of how to play

• It requires full contingent planning
  • As if you have to delegate play to “computer”
  • You would have to spell out how game should be played in every contingency
  • In chess, for example, this would be an impossible task

• In strategic form games, there is no difference between action and strategy (we will use them interchangeably)
Finite Strategy Spaces

• When $S_i$ is finite for all $i$, game is called **finite game**

• For 2 agents and small action sets, it can be expressed in matrix form

• Example: matching pennies

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• Game represents pure conflict; one player’s utility is negative other player’s utility; thus, zero sum game
Infinite Strategy Spaces

• When $S_i$ is infinite for at least one $i$, game is called infinite game

• Example: Cournot competition
  • Two firms (agents) produce homogeneous good for same market
  • Agent $i$’s action is quantity, $s_i \in [0, \infty]$, she produces
  • Agent $i$’s utility is her total revenue minus total cost
    • $u_i(s_1, s_2) = s_i p(s_1 + s_2) - c s_i$
    • $p(s)$ is price as function of total quantity, $c$ is unit cost (same for both agents)
Dominant Strategy

• Strategy $s_i \in S_i$ is **dominant strategy** for agent $i$ if
  
  $$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$
  
  for all $s'_i \in S_i$ and for all $s_{-i} \in S_{-i}$

• Example: prisoner’s dilemma

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• Action “confess” strictly dominates action “stay silent”

• Self-interested, rational behavior does not lead to socially optimal result
Dominant Strategy Equilibrium

- Strategy profile $s^*$ is (strictly) dominant strategy equilibrium if for each agent $i$, $s_i^*$ is (strictly) dominant strategy.

- Example: ISP routing game
  - ISPs share networks with other ISPs for free.
  - ISPs choose to route traffic themselves or via partner.
  - In this example, we assume cost along link is one.

<table>
<thead>
<tr>
<th>ISP 1</th>
<th>ISP 2</th>
<th>Route Yourself</th>
<th>Route via Partner</th>
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<tbody>
<tr>
<td>Route Yourself</td>
<td>(-3, -3)</td>
<td>(-6, -2)</td>
<td></td>
</tr>
<tr>
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Dominated Strategies

• Strategy $s_i \in S_i$ is strictly dominated for agent $i$ if $\exists s'_i \in S_i$:
  \[ u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i} \]

• Strategy $s_i \in S_i$ is weakly dominated for agent $i$ if $\exists s'_i \in S_i$:
  \[ u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i} \]
  \[ u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}), \exists s_{-i} \in S_{-i} \]
Rationality and Strictly Dominated Strategies

- There is no DS because of additional “suicide” strategy
  - Strictly dominated strategy for both prisoners
- No “rational” agent would choose “suicide”
  - No agent should play strictly dominated strategy

<table>
<thead>
<tr>
<th>Prisoner 2</th>
<th>Stay Silent</th>
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<th>Suicide</th>
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<tr>
<td>Stay Silent</td>
<td>(-1, -1)</td>
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<td>(0, -10)</td>
</tr>
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<td>(-1, -10)</td>
</tr>
<tr>
<td>Suicide</td>
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<td>(-10, -1)</td>
<td>(-10, -10)</td>
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Rationality and Strictly Dominated Strategies (cont.)

• If A1 knows that A2 is rational, then she can eliminate A2’s “suicide” strategy, and likewise for A2
• After one round of elimination of strictly dominated strategies, we are back to prisoner’s dilemma game
• Iterated elimination of strictly dominated strategies leads to unique outcome, “confess, confess”
• Game is dominance solvable (We will come back to this later)
How Reasonable is Dominance Solvability?

• Consider $k$-beauty contest game is dominance solvable!

\[
\begin{array}{c|c}
100 & \text{dominated} \\
(2/3)*100 & \text{dominated after removal of} \\
(2/3)*(2/3)*100 & (originally) dominated \\
\vdots & \text{strategies} \\
0 & \\
\end{array}
\]
Existence of Dominant Strategy Equilibrium

• Does matching pennies game have DSE?

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• Dominant strategy equilibria do not always exist
Best Response

- $B_i(s_{-i})$ represents agent $i$'s best response correspondence to $s_{-i}$

- Example: Cournot competition
  - $u_i(s_1, s_2) = s_ip(s_1 + s_2) - cs_i$
  - Suppose that $c = 1$ and $p(s) = \max\{0, 2 - s\}$
  - First order optimality condition gives

$$B_i(s_{-i}) = \arg\max(s_i(2 - s_i - s_{-i}) - s_i)$$

$$= \begin{cases} 
(1 - s_{-i})/2 & \text{if } s_{-i} \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

- Figure illustrates best response correspondences (functions here!)
Pure Strategy Nash Equilibrium

- (Pure strategy) Nash equilibrium is strategy profile $s^* \in S$ such that $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall i, s_i \in S_i$

- No agent can profitably deviate given strategies of others

- In Nash equilibrium, best response correspondences intersect

- Strategy profile $s^* \in S$ is Nash equilibrium iff $s_i^* \in B_i(s_{-i}^*), \forall i$
Example: Battle of the Sexes

- Couple agreed to meet this evening
- They cannot recall if they will be attending opera or football
- Husband prefers football, wife prefers opera
- Both prefer to go to same place rather than different ones
Existence of Pure Strategy Nash Equilibrium

• Does matching pennies game have pure strategy NE?

<table>
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• Pure strategy Nash equilibria do not always exist
Mixed Strategies

- Let $\Sigma_i$ denote set of probability measures over pure strategy set $S_i$
  - E.g., 45% left, 10% middle, and 45% right
- We use $\sigma_i \in \Sigma_i$ to denote mixed strategy of agent $i$, and $\sigma \in \Sigma = \prod_{i \in J} \Sigma_i$ to denote mixed strategy profile
  - This implicitly assumes agents randomize independently
- Similarly, we define $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$
- Following von Neumann-Morgenstern expected utility theory, we have
  $$u_i(\sigma) = \int_S u_i(s) d\sigma(s)$$
Strict Dominance by Mixed Strategy

- Agent 1 has no pure strategy that strictly dominates b
- However, b is strictly dominated by mixed strategy \( \left( \frac{1}{2}, 0, \frac{1}{2} \right) \)
- Action \( s_i \) is strictly dominated if there exists \( \sigma_i \) such that \( u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i} \)
- Strictly dominated strategy is never played with positive probability in mixed strategy NE
- However, weakly dominated strategies could be used in Nash equilibrium
Iterative Elimination of Strictly Dominated Strategies

• Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$

• For each agent $i$, define
  
  • $S_i^n = \{s_i \in S_i^{n-1} | \not\exists \sigma_i \in \Sigma_i^{n-1} : u_i(\sigma_i, s_i) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}^{n-1}\}$

• And define
  
  • $\Sigma_i^n = \{\sigma_i \in \Sigma_i | \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n\}$

• Finally, define $S_i^\infty$ as set of agent $i$’s strategies that survive IESDS

  • $S_i^\infty = \bigcap_{n=1}^{\infty} S_i^n$
Mixed Strategy Nash Equilibrium

• Profile \( \sigma^* \) is (mixed strategy) Nash equilibrium if for each agent \( i \)
  \[ u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall \sigma_i \in \Sigma_i \]

• Profile \( \sigma^* \) is (mixed strategy) Nash equilibrium iff for each agent \( i \)
  \[ u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*), \quad \forall s_i \in \Sigma_i \]

• Why?

• Hint: Agent \( i \)’s utility for playing mix strategies is convex combination of his utility when playing pure strategies
Mixed Strategy Nash Equilibria (cont.)

• For $G$, finite strategic form game, profile $\sigma^*$ is NE iff for each agent, every pure strategy in support of $\sigma_i^*$ is best response to $\sigma_{-i}^*$
  
  • Why?
  
  • Hint: If profile $\sigma^*$ puts positive probability on strategy that is not best response, shifting that probability to other strategies improves expected utility

• Every action in support of agent's NE mixed strategy yields same utility
Finding Mixed Strategy Nash Equilibrium

- Assume H goes to football with probability $p$ and W goes to opera with probability $q$
- Using mixed equilibrium characterization, we have

\[
p - (1 - p) = -p + 4(1 - p) \Rightarrow p = \frac{5}{7}
\]

\[
q - (1 - q) = -q + 4(1 - q) \Rightarrow q = \frac{5}{7}
\]

- Mixed strategy Nash equilibrium utilities are $\left(\frac{3}{7}, \frac{3}{7}\right)$
Example: Bertrand Competition with Capacity Constraints

- Two firms charge prices $p_1, p_2 \in [0, 1]$ per unit of same good
- There is unit demand which has to be supplied
- Customers prefer firm with lower price
- Assume each firm has capacity constraint of $2/3$ units of demand
  - If $p_1 < p_2$, firm 2 gets $1/3$ units of demand
- If both firms charge same price, each gets half of demand
- Utility of each firm is profit they make ($c = 0$, for both firms)
Example: Bertrand Competition with Capacity Constraints (cont.)

• Without capacity constraint, $p_1 = p_2 = 0$ is unique pure strategy NE
  • You will prove this in first assignment!

• With capacity constraint, $p_1 = p_2 = 0$ is no longer pure strategy NE
  • Either firm can increase its price and still have 1/3 units of demand

• We consider symmetric mixed strategy Nash equilibrium
  • I.e., both firms use same mixed strategy

• We use cumulative distribution function, $F(\cdot)$, for mixed strategies
Example: Bertrand Competition with Capacity Constraints (cont.)

• What is expected utility of firm 1 when it chooses $p_1$ and firm 2 uses mixed strategy $F(\cdot)$?

$$u_1(p_1, F(\cdot)) = F(p_1) \frac{p_1}{3} + (1 - F(p_1)) \frac{2p_1}{3}$$

• Each action in support of mixed strategy must yield same utility at NE

  • $\forall p$ in support of $F(\cdot)$

    $$\frac{2p}{3} - F(p) \frac{p}{3} = k,$$

  • $\exists k \geq 0$

    $$F(p) = 2 - \frac{3k}{p}$$
Example: Bertrand Competition with Capacity Constraints (cont.)

• Note that upper support of mixed strategy must be at \( p = 1 \), which implies that \( F(1) = 1 \)

• Combining with preceding, we obtain

\[
F(p) = \begin{cases} 
0, & \text{if } 0 \leq p \leq \frac{1}{2} \\
2 - \frac{1}{p}, & \text{if } \frac{1}{2} \leq p \leq 1 \\
1, & \text{if } p \geq 1.
\end{cases}
\]
Nash’s Theorem

• Theorem (Nash): Every finite game has mixed strategy NE

• Why is this important?
  • Without knowing the existence of equilibrium, it is difficult (perhaps meaningless) to try to understand its properties
  • Armed with this theorem, we also know that every finite game has at least one equilibrium, and thus we can simply try to locate equilibria
  • Knowing that there might be multiple equilibria, we should study efficiency/inefficiency of games’ equilibria
Example: Braess’s Paradox

- There are $2k$ drivers commuting from $s$ to $t$
- $C(x)$ indicates travel time in hours for $x$ drivers
- $k$ drivers going through $v$ and $k$ going through $w$ is NE
  - Why?
Example: Braess’s Paradox (cont.)

- Suppose we install teleportation device allowing drivers to travel instantly from $v$ to $w$
  - What is new NE? What is drivers’ commute time?
  - What is optimal commute time?
  - Does selfish routing does not minimize commute time?

- Price of Anarchy (PoA) is ratio between system performance with strategic agents and best possible system performance
  - Ratio between 2 and $3/2$ in Braess’s Paradox
Correlated Strategies

• In NE, agents randomize over strategies independently
• Agents can randomize by communicating prior to taking actions
• Example: battle of the sexes

<table>
<thead>
<tr>
<th></th>
<th>Wife</th>
<th>Football</th>
<th>Opera</th>
</tr>
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<tbody>
<tr>
<td>Husband</td>
<td>Football</td>
<td>(4, 1)</td>
<td>(-1, -1)</td>
</tr>
<tr>
<td>Opera</td>
<td>(-1, -1)</td>
<td>(1, 4)</td>
<td></td>
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• Unique mixed strategy NE is \((\left(\frac{5}{7}, \frac{2}{7}\right), \left(\frac{5}{7}, \frac{2}{7}\right))\) with utilities \((\frac{3}{7}, \frac{3}{7})\)
• Can they both do better by coordinating?
Correlated Strategies (cont.)

• Suppose there is publicly observable fair coin
• If it is heads/tails, they both get signal to go to football/opera
• If H/W sees heads, he/she believes that W/H will go to football, and therefore going to football is his/her best response
  • Similar argument can be made when he/she sees tails
• When recommendation of coin is part of Nash equilibrium, no agent has any incentives to deviate
• Expected utilities for this play of game increases to (2.5, 2.5)
Correlated Equilibrium

• Correlated equilibrium of finite game is joint probability distribution $\pi \in \Delta(S)$ such that $\forall i, s_i \in S_i$ with $\pi(s_i) > 0$, and $t_i \in S_i$

$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i}|s_i)[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})] \geq 0$$

• Distribution $\pi$ is defined to be correlated equilibrium if no agent can benefit by deviating from her recommendation, assuming other agents play according to their recommendations
Example: Game of Chicken

- (D, S) and (S, D) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing strategy will make agents strictly worse off
Example: Game of Chicken (cont.)

• Assume D1 dodges with probability $p$ and D2 dodges with probability $q$

• Using mixed equilibrium characterization, we have

\[ p - 5(1 - p) = 0 - (1 - p) \Rightarrow p = \frac{4}{5} \]

\[ q - 5(1 - q) = 0 - (1 - q) \Rightarrow q = \frac{4}{5} \]

• Mixed strategy Nash equilibrium utilities are $\left( \frac{-1}{5}, \frac{-1}{5} \right)$, people may die!
Example: Game of Chicken (cont.)

• Is this correlated equilibrium?
• If D1 gets signal to dodge
  • Conditional probability that D2 dodges is \( \frac{0.2}{0.2+0.4} = \frac{1}{3} \)
  • Expected utility of dodging is \( \left(\frac{2}{3}\right) \times (-1) \)
  • Expected utility of going straight is \( \left(\frac{1}{3}\right) \times 1 + \left(\frac{2}{3}\right) \times (-5) = -3 \)
  • Following recommendation is better
• If D1 gets signal to go straight, she knows that D2 is told to dodge, so again, D1 wants to follow recommendation
• Similar analysis works for D2, so nobody dies!
• Expected utilities increase to \((0,0)\)
Characterization of Correlated Equilibrium

• Proposition
  • Joint distribution $\pi \in \Delta(S)$ is correlated equilibrium of finite game iff
  \[ \sum_{s_{-i} \in S_{-i}} \pi(s)[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})] \geq 0, \quad \forall i, s_i, t_i \in S_i \]

• Proof
  • By definition of conditional probability, correlated equilibrium can be written as
  \[ \sum_{s_{-i} \in S_{-i}} \frac{\pi(s_{i, s_{-i}})}{\sum_{s_{-i} \in S_{-i}} \pi(s_{i, t_{-i}})} [u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})] \geq 0, \forall i, s_i \in S_i \text{ with } \pi(s_i) > 0, \text{ and } t_i \]

  • Denominator does not depend on variable of sum, so it can be factored and cancelled
  • If $\pi(s_i) = 0$, then LHS of Proposition is zero regardless of $i$ and $t_i$, so equation always holds
Questions?
Acknowledgement

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  • Vincent Conitzer [Duke CPS 590.4]