Outline

• Perfect information extensive form games
• Subgame perfect equilibrium
• Backward induction
• One-shot deviation principle
• Imperfect information extensive form games

• Readings:
  • MAS Sec. 5, GT Sec. 3 (skim through Sec. 3.4 and 3.6), Sec. 4.1, and Sec 4.2
Extensive Form Games

• So far, we have studied strategic form games
  • Agents take actions once and simultaneously

• Next, we study extensive form games
  • Agents sequentially make decisions in multi-stage games
  • Some agents may move simultaneously at some stage
  • Extensive form games can be conveniently represented by game trees
Example: Entry Deterrence Game

• Entrant chooses to enter market or stay out
• Incumbent, after observing entrant's action, chooses to accommodate or fight
• Utilities are given by \((x, y)\) at leaves for each action profile (or history)
  • \(x\) denotes utility of agent 1 (entrant) and \(y\) denotes utility of agent 2 (incumbent)
Example: Investment in Duopoly

• Agent 1 chooses to invest or not invest
• After that, both agents engage in Cournot competition
  • If agent 1 invests, then they engage in Cournot game with \( c_1 = 0 \) and \( c_2 = 2 \)
  • Otherwise, they engage in Cournot game with \( c_1 = c_2 = 2 \)
Finite Perfect-Information Extensive Form Games

Formally, each game is tuple $G = (I, (A_i)_{i \in I}, H, Z, \alpha, (\beta_i)_{i \in I}, \rho, (u_i)_{i \in I})$

- $I$ is finite set of agents
- $A_i$ is set of actions available to agent $i$
- $H$ is set of choice nodes (internal nodes of game tree)
- $Z$ is set of terminal nodes (leaves of game tree)
- $\alpha : H \mapsto 2^I$ is agent function, which assigns to each choice node set of agents
- $\beta_i : H \mapsto 2^{A_i}$ is action function, which maps choice nodes to set of actions available to agent $i$
- $\rho : H \times A \mapsto H \cup Z$ is successor function, which maps choice nodes and action profiles to new choice or terminal node, such that if $\rho(h_1, a_1) = \rho(h_2, a_2)$, then $h_1 = h_2$ and $a_1 = a_2$
- $u_i : Z \mapsto \mathbb{R}$ is utility function, which assigns real-valued utility to agent $i$ at terminal nodes
History in Extensive Form Games

- Let $H^k = \{h^k\} \subseteq \mathcal{H} \cup \mathcal{Z}$ be set of all possible stage $k$ nodes in game's tree
  - $h^0 = \emptyset$  
    initial history
  - $a^0 = (a_i^0)_{i \in a(h^0)}$  
    stage 0 action profile
  - $h^1 = (a^0)$  
    history after stage 0
  - $a^1 = (a_i^1)_{i \in a(h^1)}$  
    stage 1 action profile
  - $h^2 = (a^0, a^1)$  
    history after stage 1
  - $\vdots$
  - $h^k = (a^0, \ldots, a^{k-1})$  
    history after stage $k - 1$

- If number of stages is finite, then game is called finite horizon game

- In perfect information extensive form games, each choice (and terminal) node is associated with unique history and vice versa
Strategies in Extensive Form Games

• Pure strategies for agent $i$ is defined as contingency plan for every choice node that agent $i$ is assigned to

$$ s_i \in S_i = \prod_{h \in H : i \in \alpha(h)} \beta_i(h) $$

• Example:
  • Agent 1’s strategies: $s_1 \in S_1 = \{C, D\}$
  • Agent 2’s strategies: $s_2 \in S_2 = \{EG, EH, FG, FH\}$
  • For strategy profile $s = (C, EG)$, outcome is terminal node $\{C, E\}$
Randomized Strategies in Extensive Form Games

- **Mixed strategy**: randomizing over pure strategies
- **Behavioral strategy**: randomizing at each choice node

**Example:**
- Give behavioral strategy for agent 1
  - L with probability 0.2 and L with probability 0.5
- Give mixed strategy for agent 1 that is not behavioral strategy
  - LL with probability 0.4 and RR with probability 0.6 (why this is not behavioral?)
Example: Sequential Matching Pennies

- Consider following extensive form version of matching pennies

- How many strategies does agent 2 have?
  - $s_2 \in S_2 = \{HH, HT, TH, TT\}$

- Extensive form games can be represented as normal form games

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<thead>
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<th></th>
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<th>HT</th>
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<td>(1, -1)</td>
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- What will happen in this game?
Example: Entry Deterrence Game

- Consider following extensive form game

- What is equivalent strategic form representation?

- Two pure Nash equilibrium: (In, A) and (Out, F)

- Are Nash equilibria of this game reasonable in reality?
  - (Out, F) is sustained by noncredible threat of Entrant
Subgames

• Suppose that $V_G$ represents set of all nodes in $G$’s game tree
• Subgame $G'$ of $G$ consists of one choice node and all its successors
• Restriction of strategy $s$ to subgame $G'$ is denoted by $s_{G'}$
• Subgame $G'$ can be analyzed as its own game

• Example: sequential matching pennies
  • How many subgame does this game have?
  • Given that game itself is also considered as subgame, there are three subgames
Matrix Representation of Subgames

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<tr>
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<th>L*</th>
<th>R*</th>
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Subgame Perfect Equilibrium (SPE)

• Profile $s^*$ is SPE of game $G$ if for any subgame $G'$ of $G$, $s^*_{G'}$ is NE of $G'$

• Loosely speaking, subgame perfection will remove noncredible threats
  • Noncredible threats are not NE in their subgames

• How to find SPE?
  • One could find all of NE, then eliminate those that are not subgame perfect
  • But there are more economical ways of doing it
Backward Induction for Finite Games

• (1) Start from "last" subgames (choice nodes with all terminal children)
• (2) Find Nash equilibria of those subgames
• (3) Turn those choice nodes to terminal nodes using NE utilities
• (4) Go to (1) until no choice node remains

• [Theorem] Backward induction gives entire set of SPE
• (RR, LL) and (LR, LR) are not subgame perfect equilibria because (*R, **) is not an equilibrium
• (LL, LR) is not subgame perfect because (*L, *R) is not an equilibrium, *R is not a credible threat
Example: Stackleberg Model of Competition

• Consider variant of Cournot game where firm 1 first chooses \( q_1 \), then firm 2 chooses \( q_2 \) after observing \( q_1 \) (firm 1 is Stackleberg leader)

• Suppose that both firms have marginal cost \( c \) and inverse demand function is given by \( P(Q) = \alpha - \beta Q \), where \( Q = q_1 + q_2 \), and \( \alpha > c \)

• Solve for SPE by backward induction starting firm 2’s subgame
  
  • Firm 2 chooses \( q_2 = \arg \max_{q \geq 0} (\alpha - \beta(q_1 + q) - c)q \)
    
    - \( q_2 = (\alpha - c - \beta q_1)/2\beta \)
  
  • Firm 1 chooses \( q_1 = \arg \max_{q \geq 0} (\alpha - \beta(q + (\alpha - c - \beta q)/2\beta) - c)q \)
    
    - \( q_1 = (\alpha - c)/2\beta \)
    - \( q_2 = (\alpha - c)/4\beta \)
Example: Ultimatum Game

- Two agents want to split $c$ dollars
  - 1 offers 2 some amount $x \leq c$
  - If 2 accepts, outcome is $(c - x, x)$
  - If 2 rejects, outcome is $(0, 0)$

- What is 2’s best response if $x > 0$?
  - Yes

- What is 2’s best response if $x = 0$?
  - Indifferent between Yes or No

- What are 2’s optimal strategies?
  - (a) Yes for all $x \geq 0$
  - (b) Yes if $x > 0$, No if $x = 0$
SPE of Ultimatum Game

• What is 1’s optimal strategy for each of 2’s optimal strategies?
  • For (a), 1’s optimal strategy is to offer $x = 0$
  • For (b),
    • If agent 1 offers $x = 0$, then her utility is 0
    • If she wants to offer any $x > 0$, then she must offer $\arg\max_{x>0} (c - x)$
      • This optimization does not have any optimal solution!
    • No offer of agent 1 is optimal!

• Unique SPE of ultimatum game is:

  “Agent 1 offers 0, and agent 2 accepts all offers”
Modified Ultimatum Game

• If $c$ is in multiples of cent, what are 2’s optimal strategies?
  • (a) Yes for all $x \geq 0$
  • (b) Yes if $x > 0$, No if $x = 0$

• What are 1’s optimal strategies for each of 2’s?
  • For (a), offer $x = 0$
  • For (b), offer $x = 1$ cent

• What are SPE of modified ultimatum game?
  • Agent 1 offers 0, and agent 2 accepts all offers
  • Agent 1 offers 1 cent, and agent 2 accept all offers except 0

• Show that for every $\bar{x} \in [0, c]$, there exists NE in which 1 offers $\bar{x}$
  • What is agent 2's optimal strategy?
limitation of Backward Induction

• If there are ties, how they are broken affects what happens up in tree
  • There could be too many equilibria
Example: Bargaining Game

- Two agents want to split $c = 1$ dollar
- First, 1 makes her offer
- Then, 2 decides to accept or reject
- If 2 rejects, then 2 makes **new offer**
- Then, 1 decides to accept or reject
- Let $x = (x_1, x_2)$ with $x_1 + x_2 = 1$ denote allocations in 1st round
- Let $y = (y_1, y_2)$ with $y_1 + y_2 = 1$ denote allocations in 2nd round
Backward Induction for Bargaining Game

• Second round is ultimatum game with unique SPE
  • Agent 2 offers 0, and agent 1 accepts all offers

• What is 2’s optimal strategy in her round 1’s subgame?
  • (a) If $x_2 \leq 1$, reject
  • (b) If $x_2 = 1$, accept, and reject otherwise

• What are 1’s optimal strategies in round 1 for each of 2’s?
  • For both (a) and (b), agent 1 is indifferent between all strategies
  • Agent 1’s weakly dominant strategy is to offer $x_2 = 1$

• How many SPE does this game have?
  • Infinitely many! In all SPE, agent 2 gets everything
  • Last mover’s advantage: In every SPE, agent who makes offer in last round obtains everything
Example: Discounted Bargaining Game

• Suppose utilities are discounted every round by discount factor, $0 < \delta_i < 1$

• What is unique SPE of (1)?
  • 2 offers $y_1 = 0$ and 1 accepts all offers

• What are optimal strategies in (2)?
  • (a) Yes if $x_2 \geq \delta_2$, No otherwise
  • (b) Yes if $x_2 > \delta_2$, No otherwise

• What are optimal strategies in (3)?
  • For (a), offer $x_2 = \delta_2$
  • For (b), there is no optimal strategy
Unique SPE of Discounted Bargaining Game

• What are SPE strategies?
  • Agent 1’s proposes \((1 - \delta_2, \delta_2)\)
  • Agent 2 only accepts proposals with \(x_2 \geq \delta_2\)
  • Agent 2 proposes \((0, 1)\) after any history in which 1’s proposal is rejected
  • Agent 1 accepts all proposals of Agent 2

• What is SPE outcome of game?
  • Agent 1 proposes \((1 - \delta_2, \delta_2)\)
  • Agent 2 accepts
  • Resulting utilities are \((1 - \delta_2, \delta_2)\)

• Desirability of earlier agreement yields positive utility for agent 1
Stahl’s Bargaining Model (for Finite Horizon Games)

- 2 rounds: \((1 − \delta_2)\)
- 3 rounds: \((1 − \delta_2) + \delta_1 \delta_2\)
- 4 rounds: \((1 − \delta_2)(1 + \delta_1 \delta_2)\)
- 5 rounds: \((1 − \delta_2)(1 + \delta_1 \delta_2) + \delta_1 \delta_2\)
- 2k rounds: \((1 − \delta_2) \left( \frac{1-(\delta_1 \delta_2)^k}{1-\delta_1 \delta_2} \right)\)
- 2k+1 rounds: \((1 − \delta_2) \left( \frac{1-(\delta_1 \delta_2)^k}{1-\delta_1 \delta_2} \right) + (\delta_1 \delta_2)^k\)

- Taking limit as \(k \to \infty\), we see that agent 1 gets \(x_1^* = \frac{1−\delta_2}{1−\delta_1 \delta_2}\) at SPE
Rubinstein’s Infinite Horizon Bargaining Model

- Suppose agent can alternate offers forever
- There are two types of outcome to consider
  - At round $t$, one agent accepts her offer $(x^1, \text{No}, x^2, \text{No}, \ldots, x^t, \text{Yes})$
  - Every offer gets rejected: $(x^1, \text{No}, x^2, \text{No}, \ldots, x^k, \text{No}, \ldots)$
- This is not finite horizon game, backward induction cannot be used
- We need different method to verify any SPE
One-Shot Deviation Principle

• One-shot deviation from strategy $s$ means deviating from $s$ in single stage and conforming to it thereafter.

• Strategy profile $s^*$ is SPE if and only if there exists no profitable one-shot deviation for each subgame and every agent.

• This follows from principle of optimality of dynamic programming.
SPE for Rubinstein’s Model

- Recall that in Stahl’s model, for $k \to \infty$, $x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$

- Is following strategy profile $s^*$ SPE?
  - Agent 1 proposes $x^*$ and accepts $y$ if and only if $y \geq y_1^*$
  - Agent 2 proposes $y^*$ and accepts $x$ if and only if $x \geq x_2^*$
  - $x^* = (x_1^*, x_2^*)$, $x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$, $x_2^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$
  - $y^* = (y_1^*, y_2^*)$, $y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$, $y_2^* = \frac{1-\delta_1}{1-\delta_1\delta_2}$
One-Shot Deviation Principle for Rubinstein's Model

- First note that this game has two types of subgames
  - (1) first move is offer
  - (2) first move is response to offer

- For (1), suppose offer is made by agent 1
  - If agent 1 adopts $s^*$, agent 2 accepts, agent 1 gets $x_1^*$
  - If agent 1 offers $> x_2^*$, agent 2 accepts, and agent 1 gets $x_1 < x_1^*$
  - If agent 1 offers $< x_2^*$, agent 2 rejects and offers $y_1^*$, agent 1 accepts and gets $\delta y_1 < x_1^*$

- For (2), suppose agent 1 is responding to offer $y_1 \geq y_1^*$
  - If agent 1 adopts $s^*$, she accepts and gets $y_1$
  - If agent 1 rejects and offers $x_2^*$ in next round, agent 2 accepts, agent 1 gets $\delta x_1 = y_1^* \leq y_1$

- For (2), suppose agent 1 is responding to offer $y_1 < y_1^*$
  - If agent 1 adopts $s^*$, she rejects and offers $x_2^*$ in next round, agent 2 accepts, agent 1 gets $\delta x_1 = y_1^* > y_1$
  - If agent 1 accepts, she gets $y_1 < y_1^*$

- Hence $s^*$ is SPE (in fact unique SPE, check GT, Section 4.4.2 to verify)
Rubinstein’s Model for Symmetric Agents

• Suppose that $\delta_1 = \delta_2$
  • If agent 1 moves first, division is $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$
  • If agent 2 moves first, division is $\left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$

• First mover’s advantage is related to impatience of agents
  • If $\delta \to 1$, FMA disappears and outcome tends to $\left(\frac{1}{2}, \frac{1}{2}\right)$
  • If $\delta \to 0$, FMA dominates and outcome tends to $(1,0)$
Imperfect Information Extensive Form Games

• In perfect information games, agents know choice nodes they are in
  • Agents know all prior actions
  • Recall that in such games choice nodes are equal to histories that led to them
• Agents may have partial or no knowledge of actions taken by others
• Agents may also have imperfect recall of actions taken by themselves
Example: Imperfect Information Sequential Matching Pennies

- Agent 1 takes action
- Agent 2 does not see agent 1’s action
- Agent 2 takes action, and outcome is revealed

- Information set is collection of choice nodes that cannot be distinguished by agents whose turn it is
- Set of agents and their actions at each choice node in information set has to be the same, otherwise, agents could distinguish between nodes
Finite Imperfect-Information Extensive Form Games

• Formally, each game is tuple \( G = (I, (A_i)_{i \in I}, \mathcal{H}, Z, \alpha, (\beta_i)_{i \in I}, \rho, (u_i)_{i \in I}, I) \)

  • \( (I, (A_i)_{i \in I}, \mathcal{H}, Z, \alpha, (\beta_i)_{i \in I}, \rho, (u_i)_{i \in I}) \) is perfect information, extensive form game

  • \( I = (I_1, \ldots, I_n) \), where \( I_j = \{h_{j,1}, \ldots, h_{j,k_j}\} \), is partition of \( \mathcal{H} \) such that if \( h, h' \in I_j \), then 
    \[
    \alpha(h) = \alpha(h'), \text{ and for all } i \in \alpha(h), \beta_i(h) = \beta_i(h')
    \]
Example: Poker-Like Game

- What are agent 1’s strategies?
  - \{RR, RC, CR, CC\}

- What are agent 2’s strategies?
  - \{CC, CF, FC, FF\}

- How can we find NE of this game?
  - Model game as normal form zero-sum game
    - Each cell represents expected utilities (nature’s coin toss)
  - Eliminated (weakly) dominated strategies
  - Solve for (mixed strategy) NE
Example: Kune Poker

https://justinsermo.com/posts/cfr/
Imperfect Recall, Mixed vs Behavioral Strategies

• Consider mixed strategies
  • What is NE of this game?
    • \((R,D)\) with outcome utilities \((2,2)\)

• Consider behavioral strategies
  • What is 1’s expected utility if she does \(p, 1 - p\)
    • \(p^2 + 100p(1 - p) + 2(1 - p)\)
  • What is 1’s best response?
    • \(p = \frac{98}{198}\)
  • What is NE of this game?
    • \(\left(\frac{98}{198}, \frac{100}{198}\right), (0,1)\)
Solving Extensive Form Games: Perfect vs Imperfect Information

• In perfect information games, optimal strategy for each subgame can be determined by that subgame alone (how backward induction works!)
  • We can forget how we got here
  • We can ignore rest of game

• In imperfect information games, this is not necessarily true
  • We cannot forget about path to current node
  • We cannot ignore other subgames
Example

• Is always accommodating good strategy?
  • No, leads to utility of -2.5 for incumbent

• Is always fighting good strategy?
  • No, leads to utility of -1.5 for incumbent

• What should incumbent do?
  • A with 3/8 probability and F with 5/8

• What if we swap 2 and -2?
  • A with 7/8 probability and F with 1/8
Subgame Perfection and Imperfect Information

• There are two subgames: game itself and subgame after agent 1 plays R
  • (R, RR) is NE and SPE

• But, why should 2 play R after 1 plays L/M?
  • This is noncredible threat

• There are more sophisticated equilibrium refinements that rule this out
Questions?
Acknowledgement

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- Asu Ozdaglar [MIT 6.254]
- Vincent Conitzer [Duke CPS 590.4]