

# Assignment 1

Due on Wednesday Feb. 14<sup>th</sup> (before the class)

Please read the rules for assignments on the course web page (<https://ece.uwaterloo.ca/~smzahedi/crs/ece750/>). Use Piazza (preferred) or directly contact Seyed (smzahedi@uwaterloo.ca) with any questions.

**1. Utility Functions. (10 points).** Alice is making plans for Spring Break. She most prefers to go to Cancun, a trip that would cost her \$4000. Another good option is to go to Miami, which would cost her only \$1500. Alice is really excited about Spring Break and cares about nothing else in the world right now. As a result, Alice's utility  $u$  as a function of her budget  $b$  is given by:

$$u(b) = \begin{cases} -1 & \text{if } b < \$0, \\ 0 & \text{if } \$0 \leq b < \$1500, \\ 1 & \text{if } \$1500 \leq b < \$4000, \\ 2 & \text{if } b \geq \$4000. \end{cases}$$

Alice's budget right now is \$1000 (which would give her a utility of 0).

Alice's wealthy friend Kathy is aware of Alice's predicament and wants to offer her a "fair gamble." Define a fair gamble to be a *random variable with expected value* \$0. An example of fair gamble (with two outcomes) is the following: \$-750 with probability  $2/5$ , and \$500 with probability  $3/5$ . If Alice were to accept this gamble, she would end up with \$250 with probability  $2/5$ , and with \$1500 with probability  $3/5$ . Therefore, Alice's expected utility for accepting this gamble is  $(2/5) \cdot (0) + (3/5) \cdot (1) = 3/5$ .

**a.** Find a fair gamble with two outcomes that would not change Alice's expected utility (5 points).

**b.** Find a fair gamble with two outcomes that would strictly decrease Alice's expected utility (5 points).

**2. Strict and Weak Dominance. (15 points).**

**a.** Find all Nash equilibria of the following game (10 points).

(2,3)	(6,2)	(6,2)
(4,1)	(3,2)	(0,4)
(2,5)	(3,7)	(2,6)

**b.** Construct a single  $3 \times 3$  normal-form game that simultaneously has all four of the following properties (5 points).

1. The game is not solvable by weak dominance (at least one agent does not have a weakly dominant strategy).
2. The game is solvable by iterated weak dominance (so that one pure strategy per agent remains).
3. In addition to the iterated weak dominance solution (which is a Nash equilibrium), there is a second pure-strategy Nash equilibrium.
4. Both agents strictly prefer the second equilibrium to the first.

If you cannot get all four properties, construct an example with as many of the properties as you can.

### 3. Correlated Equilibrium. (20 points).

- a. Consider the following game.

(7,7)	(1,8)
(8,1)	(-1,-1)

Find the correlated equilibrium that achieves the highest symmetric utilities (10 points).

- b. Suppose that  $f_i : A_i \rightarrow A_i$  is a function that maps recommended strategies (i.e., signals) to agent  $i$ 's playing strategies. In other words, suppose that upon receiving a recommendation to play  $r_i$ , agent  $i$  plays  $f_i(r_i)$ . Using one of the definitions provided in the slides, prove that a joint distribution  $\pi \in \Delta(A)$  is a correlated equilibrium of a finite game **if and only if**

$$\sum_{r \in A} \pi(r) [u_i(r_i, r_{-i}) - u_i(f_i(r_i), r_{-i})] \geq 0,$$

for all  $i$  and for all functions  $f_i : A_i \rightarrow A_i$  (10 points).

### 4. (Pure and Mixed Nash Equilibrium. (30 points).

- a. Each of  $n$  candidates chooses a position to take on the real line in the interval  $[0, I]$ . There is a continuum of citizens, whose favourite positions are uniformly distributed between  $[0, I]$ . A candidate attracts votes of citizens whose favourite positions are closer to his position than to the position of any other candidate; if  $k$  candidates choose the same position, then each receives the fraction  $1/k$  of the votes that the position attracts. The pay-off of each candidate is his vote share.

- Model the game and find all pure strategy Nash equilibria when  $n = 2$  (10 points).
- Show that there does not exist a pure strategy Nash equilibrium when  $n = 3$  (No points, just for fun!).
- Find a mixed strategy Nash equilibrium when  $n = 3$  (No points, just for fun!).

**b.** Two agents are involved in a dispute over an object. The value of the object to agent  $i$  is  $v_i > 0$ . Time is modelled as a continuous variable that starts at 0 and runs indefinitely. Each agent chooses when to concede the object to the other agent; if the first agent to concede does so at time  $t$ , the other agent obtains the object at that time. If both agents concede simultaneously, the object is split equally between them, agent  $i$  receiving a utility of  $v_i/2$ . Time is valuable: until the first concession each agent loses one unit of utility per unit of time.

- Formulate this situation as a strategic game (3 points).
  - Show that in all Nash equilibria one of the agents concedes immediately (7 points).
- c.** Consider the following game.

(1,0,3)	(1,0,1)	(3,1,2)	(1,0,1)	(0,5,0)	(1,0,5)
(1,0,4)	(0,3,0)	(1,0,1)	(3,1,2)	(0,1,2)	(0,1,3)

Agents 1, 2, and 3 pick the row, column, and matrix respectively. Find a mixed strategy Nash equilibrium that is not a pure-strategy Nash equilibrium (10 points).

**5. Cournot Competition. (15 points).** Consider the following algorithm for the iterative elimination of (strictly) dominated strategies:

1. Let  $A_i^0 = A_i$  and  $S_i^0 = S_i$  for all  $i \in N$ , where  $N$  is the set of agents,  $A_i$  is the set of agent  $i$ 's actions, and  $S_i$  is the set of agent  $i$ 's mixed strategies.
2. Set  $k = 1$ .
3. For each agent  $i$ , define:

$$A_i^k = \{a_i \in A_i^{k-1} \mid \nexists s_i \in S_i^{k-1} : u_i(s_i, a_{-i}) > u_i(a_i, a_{-i}) \forall a_{-i} \in A_{-i}^{k-1}\}.$$

4. Define:

$$S_i^k = \{s_i \in S_i \mid s_i(a_i) > 0 \text{ only if } a_i \in A_i^k\}$$

5.  $k = k + 1$ .
6. Go to (3).

Now, consider a Cournot competition between two firms. **a.** Suppose that  $c = 2$  and  $p(a) = \max\{0, 4 - 2a\}$ . Run iterative elimination of strictly dominated strategy and calculate  $A_i^k$  for  $i = \{1, 2\}$  and  $k = \{0, 1, 2, \infty\}$  (5 points).

**b.** Now, suppose that  $c = 1$  and  $p(a) = \max\{0, 4 - a\}$ . Find the Nash equilibrium strategies and utilities (5 points).

**c.** In part **b.**, what are optimal symmetric utilities if, instead of competing, the two firms cooperate (5 points)?

**6. Bertrand Competition. (10 points).** Consider a Bertrand competition between two firms. Firms charge prices  $p_1, p_2 \in [0, P]$  per each unit of same

item. Customers prefer the firm with the lower price. If both firms charge the same price, each gets half of customers. Utility of each firm is profit they make minus the costs. Suppose  $Q(p)$  denotes demand at price  $p$  and  $c$  is marginal cost per unit for both firms.

- a.** Model the game as a normal form game (3 points).
- b.** Find all Nash equilibria of the game (7 points).