1. Consider the following game. There are two agents, who each select \( a_i \geq 0 \). The utility of agent \( i \) is 
\[ u_i(a_i, a_j) = \sqrt{a_i + qa_j} - a_i, \]
where \( q > -1 \) measures the influence of agent \( j \)'s action on agent \( i \)'s utility.

a. (10 points) Calculate agent \( i \)'s best response function. Is it increasing or decreasing in \( a_j \)?

b. (10 points) Find the unique symmetric Nash equilibria of the game.

c. (Bonus Question - 10 points) Are there other, non-symmetric, equilibria? (hint: the answer depends on \( q \).)

2. There are two (and only two) firms in a market, making identical products. Each firm decides on a quantity \( q_i \in [0, 5] \). The price of the product is determined by the total production as 
\[ P = \max\{6 - (q_1 + q_2), 0\}. \]
Each firm’s costs are zero, so wishes to maximize profit: 
\[ v_i = q_i P. \]

a. (10 points) Suppose the firms make their choices simultaneously. Find the unique NE.

b. (10 points) Suppose that firm 1 first chooses \( q_1 \), which firm 2 sees and then selects \( q_2 \). Is the NE outcome from part (a) still a NE outcome of this extensive-form game?

c. (10 points) What is the subgame perfect equilibrium? Explain any difference with solution(s) for (b).

3. Consider the following game. There are \( n \geq 2 \) agents. Each one simultaneously decides whether to undertake some action, \( g \), or not to. If no one chooses \( g \), each agent’s utility is zero. If one or more agents choose \( g \), everyone gets a benefit of \( x > 1 \), and, additionally, each agent \( i \) choosing \( a_i = g \) bears a cost of 1.

a. (5 points) Formulate the game.

b. (15 points) Find a symmetric mixed-strategy equilibrium in which each agent chooses \( g \) with probably \( p^* \). To do so:
   - Formulate the "indifference condition" needed for agent \( i \) to be willing to play a mixed strategy.
   - Solve the indifference condition to get \( p^* \) as it depends on \( x \) and \( n \).

4. (Bonus Question) There are three agents bargaining over the split of a dollar. The protocol is as follows. At the beginning of period 1, agent \( i \) is recognized to be the proposer with probability \( p_i \), where \( p_1 + p_2 + p_3 = 1 \) and \( p_1 < p_2 < p_3 \). The proposer makes a proposal \( (x_1, x_2, x_3) \), such that the amount proposed for \( j \) is \( x_j \geq 0 \) for all \( j \) and \( x_1 + x_2 + x_3 \leq 1 \). Then the agents vote on the proposal: if it receives two or more votes in favour (assume the proposer is bound to vote in favour), the proposal passes and the split is implemented; if not then we move to period 2. Period 2 follows the same procedure as period 1 (the probability of being the proposer is independent across the two periods). However, period 2 is the last period; if the proposal receives less than two votes in favour, all agents get utility 0. Each agent proportionally discounts utilities in period 2 by \( \delta \in [0, 1] \).

a. (10 points) Find the subgame perfect equilibrium of the game.

b. (10 points) Calculate the expected utility for each agent before the game begins.