Lecture 3: Games and Solution Concepts (I)

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Outline

• Pure strategies, existence and examples
• Mixed strategies
• Mixed strategy Nash equilibrium

• Readings:
  • MAS Sec. 3.2 and 3.4, GT Sec. 1 and 2
Second Price Auction (Com. Info.)

• There is one object for sale
• Agent $i$’s valuation is $v_i$, and assume
  • $v_1 > v_2 > \cdots > 0$
  • Everybody knows all valuations (we will see incomplete inf. later)
• Agent $i$ submit her bid, $b_i$, simultaneously with other agents
• Agent with highest bid wins, and pays second highest bid
• Agent $i$’s profit is $v_i - b_j$ if she wins, and 0 otherwise
Truthful Bidding

• **Proposition**: In second price auction, **truthful bidding**, i.e., $b_i = v_i$ for all $i$, is Nash equilibrium

• **Proof**:
  • If everyone bids truthfully, does winner have incentives to change her bid?
  • If everyone bids truthfully, does looser have incentives to change her bid?

• Are there other Nash equilibria?
  • Is $(v_1, 0, ..., 0)$ also Nash equilibrium? What about $(v_1, v_2, 0, ..., 0)$?
Truthful Bidding (cont.)

• Truthful equilibrium is **Weakly Dominant Nash Equilibrium**
  
  • Truthful bidding \( i.e., b_i = v_i \) weakly dominates all other strategies

• Picture proof:
  
  • Suppose \( B_i^* = \max_{j \neq i} b_j \) represents maximum bids excluding \( i \)'s bid
Truthful Bidding (cont.)

• First graph shows payoff for bidding $b_i = v_i$

• Second graph shows payoff for bidding $b_i < v_i$
  • Notice that whenever $b_i \leq B_i^* \leq v_i$, then $i$ receives utility 0

• Third graph shows payoff for bidding $b_i > v_i$
  • Notice that whenever $v_i < B_i^* < b_i$, then $i$ receives negative utility

• Implication: other equilibria involve weakly dominated strategies
Nonexistence of Pure Strategy Nash Equilibria (cont.)

• Example: Penalty kick game

<table>
<thead>
<tr>
<th></th>
<th>Kicker Left</th>
<th>Kicker Middle</th>
<th>Kicker Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goalkeeper Left</td>
<td>(1, -1)</td>
<td>(-2, 2)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>Goalkeeper Middle</td>
<td>(-2, 2)</td>
<td>(2, -2)</td>
<td>(-2, 2)</td>
</tr>
<tr>
<td>Goalkeeper Right</td>
<td>(-1, 1)</td>
<td>(-2, 2)</td>
<td>(1, -1)</td>
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</table>

• How would you play this game?
  • Would you always show up left? Would this be a “good strategy”?

• Empirical and experimental evidence suggests that most players “randomize” between their actions
Mixed Strategies

• Let $\Sigma_i$ denote set of **probability measures** over pure strategy (action) set $S_i$
  - E.g., 45% left, 10% middle, and 45% right

• We use $\sigma_i \in \Sigma_i$ to denote mixed strategy of player $i$, and $\sigma \in \Sigma = \prod_{i \in J} \Sigma_i$ to denote **mixed strategy profile**
  - This implicitly assumes agents **randomize independently**

• Similarly, we define $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$

• Following von Neumann-Morgenstern expected utility theory, we have

$$u_i(\sigma) = \int_S u_i(s) d\sigma(s)$$
Strict Dominance by Mixed Strategy

- Agent 1 has no pure strategy that strictly dominates b.
- However, b is strictly dominated by mixed strategy $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$.
- Action $s_i$ is strictly dominated if there exists $\sigma_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$, $\forall s_{-i} \in S_{-i}$.
- Strictly dominated strategy is never played with positive probability in mixed strategy Nash equilibrium.
- However, weakly dominated strategies could be used in Nash equilibrium (e.g., second price auction).

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>a</th>
<th>b</th>
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</thead>
<tbody>
<tr>
<td>a</td>
<td>(2, 0)</td>
<td>(-1, 0)</td>
</tr>
<tr>
<td>b</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>c</td>
<td>(-1, 0)</td>
<td>(2, 0)</td>
</tr>
</tbody>
</table>
Iterative Elimination of Strictly Dominated Strategies (Revisited)

• Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$

• For each agent $i$, define
  
  $S_i^n = \{s_i \in S_i^{n-1} | \not\exists \sigma_i \in \Sigma_i^{n-1}: u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}^{n-1}\}$

• And define
  
  $\Sigma_i^n = \{\sigma_i \in \Sigma_i | \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n\}$

• Finally, define $S_i^\infty$ as set of agent $i$’s strategies that survive IESDS
  
  $S_i^\infty = \bigcap_{n=1}^{\infty} S_i^n$
Mixed Strategy Nash Equilibrium

• Mixed Nash Equilibrium
  • Profile $\sigma^*$ is (mixed strategy) Nash equilibrium if for each agent $i$
    \[
    u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall \sigma_i \in \Sigma_i
    \]

• Proposition: Profile $\sigma^*$ is (mixed strategy) Nash equilibrium iff for each agent $i$
  \[
  u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*), \quad \forall s_i \in \Sigma_i
  \]

• Hint: Agent $i$’s utility for playing mix strategies is convex combination of his utility when playing pure strategies
• **Proposition**: For $G$, finite strategic form game, profile $\sigma^*$ is Nash equilibrium iff for each agent, every pure strategy in support of $\sigma^*_i$ is best response to $\sigma^*_{-i}$

• **Proof idea**: If mixed strategy profile is putting positive probability on strategy that is not best response, then shifting that probability to other strategies would improve expected utility

• **Implication**: Every action in support of any agent’s equilibrium mixed strategy yields same utility

• This extends to infinite games as well!
Examples

- Example: Matching pennies
  - Unique mixed strategy Nash equilibrium is \( \left( \frac{1}{2}, \frac{1}{2} \right) \)

- Example: Battle of sexes
  - Two pure strategy Nash equilibria and one mixed strategy \( \left( \left( \frac{5}{7}, \frac{2}{7} \right), \left( \frac{2}{7}, \frac{5}{7} \right) \right) \)
Finding Mixed Strategy Nash Equilibrium

- Assume $H$ goes to football with prob. $p$ and $W$ goes to opera with prob. $q$
- Using mixed equilibrium characterization, we have

\[
P - (1 - P) = -p + 4(1 - p) \Rightarrow p = \frac{5}{7}
\]

\[
q - (1 - q) = -q + 4(1 - q) \Rightarrow q = \frac{5}{7}
\]

- Mixed strategy Nash equilibrium utilities are \(\left(\frac{3}{7}, \frac{3}{7}\right)\)
Bertrand Competition with Capacity Constraints

• Two firms charge prices $p_1, p_2 \in [0, 1]$ per unit of same good
• There is unit demand and customers choose firm with lower price
• If both firms charge same price, each firm gets half demand
• All demand has to be supplied
• Utility of each firm is profit they make
  • We assume for simplicity that cost is equal to 0 for both firms
Bertrand Competition with Capacity Constraints (cont.)

• We showed $p_1 = p_2 = 0$ is unique pure strategy Nash equilibrium.
• Assume each firm has capacity constraint of 2/3 units of demand.
• This implies that when $p_1 < p_2$, firm 2 gets 1/3 units of demand.
• $p_1 = p_2 = 0$ is no longer pure strategy Nash equilibrium.
  • Either firm can increase its price and still have 1/3 units of demand.
• What is (mixed) strategy Nash equilibrium?
Bertrand Competition with Capacity Constraints (cont.)

• We consider symmetric strategies (i.e., both firms use same mixed strategy)

• We use cumulative distribution function $F(.)$ to represent mixed strategy used by either firm

• What is expected utility of firm 1 when it chooses $p_1$ and firm 2 uses mixed strategy $F(.)$?

$$u_1(p_1, F(.)) = F(p_1) \frac{p_1}{3} + (1 - F(p_1)) \frac{2p_1}{3}$$

• Knowing that each action in support of mixed strategy must yield same utility at equilibrium, we obtain for all $p$ in support of $F(.)$

$$-F(p) \frac{p}{3} + \frac{2p}{3} = k,$$

for some $k \geq 0$, which leads to

$$F(p) = 2 - \frac{3k}{p}$$
Bertrand Competition with Capacity Constraints (cont.)

- Note that upper support of mixed strategy must be at $p = 1$, which implies that $F(1) = 1$
- Combining with preceding, we obtain

$$F(p) = \begin{cases} 
0, & \text{if } 0 \leq p \leq \frac{1}{2} \\
2 - \frac{1}{p}, & \text{if } \frac{1}{2} \leq p \leq 1 \\
1, & \text{if } p \geq 1.
\end{cases}$$
Questions?
Acknowledgement

• This lecture is a slightly modified version of one prepared by
  • Asu Ozdaglar [MIT 6.254]