ECE700.07: Game Theory with Engineering Applications

Lecture 3: Games and Solution Concepts (I)

Seyed Majid Zahedi
Outline

• Pure strategies, existence and examples
• Mixed strategies
• Mixed strategy Nash equilibrium

• Readings:
  • MAS Sec. 3.2 and 3.4, GT Sec. 1 and 2
Second Price Auction (Com. Info.)

- There is one object for sale
- Agent $i$’s valuation is $v_i$, and assume
  - $v_1 > v_2 > \cdots > 0$
  - Everybody knows all valuations (we will see incomplete inf. later)
- Agent $i$ submit her bid, $b_i$, \textit{simultaneously} with other agents
- Agent with highest bid \textit{wins}, and pays \textit{second} highest bid
- Agent $i$’s profit is $v_i - b_j$ if she wins, and 0 otherwise
Truthful Bidding

• **Proposition**: In second price auction, *truthful bidding*, i.e., $b_i = v_i$ for all $i$, is Nash equilibrium

• **Proof**:
  • If everyone bids truthfully, does winner have incentives to change her bid?
  • If everyone bids truthfully, does looser have incentives to change her bid?

• Are there other Nash equilibria?
  • Is $(v_1, 0, ..., 0)$ also Nash equilibrium? What about $(v_1, v_2, 0, ..., 0)$?
Truthful Bidding (cont.)

- Truthful equilibrium is \textbf{Weakly Dominant Nash Equilibrium}
  - Truthful bidding (\textit{i.e.}, $b_i = v_i$) weakly dominates all other strategies
- Picture proof:
  - Suppose $B_i^* = \max_{j \neq i} b_j$ represents maximum bids excluding $i$’s bid
Truthful Bidding (cont.)

• First graph shows payoff for bidding $b_i = v_i$

• Second graph shows payoff for bidding $b_i < v_i$
  • Notice that whenever $b_i \leq B_i^* \leq v_i$, then $i$ receives utility 0

• Third graph shows payoff for bidding $b_i > v_i$
  • Notice that whenever $v_i < B_i^* < b_i$, then $i$ receives negative utility

• Implication: other equilibria involve weakly dominated strategies
Nonexistence of Pure Strategy Nash Equilibria (cont.)

- Example: Penalty kick game

<table>
<thead>
<tr>
<th>Goalkeeper</th>
<th>Kicker</th>
<th>Left</th>
<th>Middle</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>(1, -1)</td>
<td>(-2,2)</td>
<td>(-1, 1)</td>
<td></td>
</tr>
<tr>
<td>Middle</td>
<td>(-2,2)</td>
<td>(2, -2)</td>
<td>(-2,2)</td>
<td></td>
</tr>
<tr>
<td>Right</td>
<td>(-1, 1)</td>
<td>(-2,2)</td>
<td>(1, -1)</td>
<td></td>
</tr>
</tbody>
</table>

- How would you play this game?
  - Would you always show up left? Would this be a “good strategy”?

- Empirical and experimental evidence suggests that most players “randomize” between their actions
Mixed Strategies

• Let $\Sigma_i$ denote set of **probability measures** over pure strategy (action) set $S_i$
  • E.g., 45% left, 10% middle, and 45% right

• We use $\sigma_i \in \Sigma_i$ to denote mixed strategy of player $i$, and $\sigma \in \Sigma = \prod_{i \in J} \Sigma_i$ to denote **mixed strategy profile**
  • This implicitly assumes agents **randomize independently**

• Similarly, we define $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$

• Following von Neumann-Morgenstern expected utility theory, we have

\[
u_i(\sigma) = \int_{S} u_i(s) d\sigma(s)\]
Strict Dominance by Mixed Strategy

- Agent 1 has no pure strategy that strictly dominates b.
- However, b is strictly dominated by mixed strategy \( \left( \frac{1}{2}, 0, \frac{1}{2} \right) \).
- Action \( s_i \) is strictly dominated if there exists \( \sigma_i \) such that \( u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i} \).
- Strictly dominated strategy is never played with positive probability in mixed strategy Nash equilibrium.
- However, weakly dominated strategies could be used in Nash equilibrium (e.g., second price auction).

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>(2, 0)</td>
<td>(-1, 0)</td>
</tr>
<tr>
<td>b</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>c</td>
<td>(-1, 0)</td>
<td>(2, 0)</td>
</tr>
</tbody>
</table>
Iterative Elimination of Strictly Dominated Strategies (Revisited)

• Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$

• For each agent $i$, define

$$S_i^n = \{s_i \in S_i^{n-1} \mid \nexists \sigma_i \in \Sigma_i^{n-1}: u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}^{n-1}\}$$

• And define

$$\Sigma_i^n = \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) > 0 \ \text{only if} \ s_i \in S_i^n\}$$

• Finally, define $S_i^\infty$ as set of agent $i$’s strategies that survive IESDS

$$S_i^\infty = \bigcap_{n=1}^{\infty} S_i^n$$
Mixed Strategy Nash Equilibrium

• Mixed Nash Equilibrium
  
  • Profile $\sigma^*$ is (mixed strategy) Nash equilibrium if for each agent $i$
    \[ u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}), \quad \forall \sigma_i \in \Sigma_i \]

• Proposition: Profile $\sigma^*$ is (mixed strategy) Nash equilibrium iff for each agent $i$
  \[ u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(s_i, \sigma^*_{-i}), \quad \forall s_i \in \Sigma_i \]

• Hint: Agent $i$’s utility for playing mix strategies is convex combination of his utility when playing pure strategies
Mixed Strategy Nash Equilibria (cont.)

- **Proposition**: For $G$, finite strategic form game, profile $\sigma^*$ is Nash equilibrium iff for each agent, every pure strategy in support of $\sigma^*_i$ is best response to $\sigma^*_{-i}$

- **Proof idea**: If mixed strategy profile is putting positive probability on strategy that is not best response, then shifting that probability to other strategies would improve expected utility

- **Implication**: Every action in support of any agent’s equilibrium mixed strategy yields same utility

- This extends to infinite games as well!
Examples

- Example: Matching pennies
  - Unique mixed strategy Nash equilibrium is \( \left( \frac{1}{2}, \frac{1}{2} \right) \)

- Example: Battle of sexes
  - Two pure strategy Nash equilibria and one mixed strategy \( \left( \frac{5}{7}, \frac{2}{7}, \frac{2}{7}, \frac{5}{7} \right) \)
Finding Mixed Strategy Nash Equilibrium

- Assume H goes to football with prob. $p$ and W goes to opera with prob. $q$
- Using mixed equilibrium characterization, we have
  \[
  p - (1 - p) = -p + 4(1 - p) \implies p = \frac{5}{7}
  \]
  \[
  q - (1 - q) = -q + 4(1 - q) \implies q = \frac{5}{7}
  \]
- Mixed strategy Nash equilibrium utilities are \( \left( \frac{3}{7}, \frac{3}{7} \right) \)
Bertrand Competition with Capacity Constraints

• Two firms charge prices $p_1, p_2 \in [0, 1]$ per unit of same good
• There is unit demand and customers choose firm with lower price
• If both firms charge same price, each firm gets half demand
• All demand has to be supplied
• Utility of each firm is profit they make
  • We assume for simplicity that cost is equal to 0 for both firms
Bertrand Competition with Capacity Constraints (cont.)

- We showed \( p_1 = p_2 = 0 \) is unique pure strategy Nash equilibrium
- Assume each firm has capacity constraint of 2/3 units of demand
- This implies that when \( p_1 < p_2 \), firm 2 gets 1/3 units of demand
- \( p_1 = p_2 = 0 \) is no longer pure strategy Nash equilibrium
  - Either firm can increase its price and still have 1/3 units of demand
- What is (mixed) strategy Nash equilibrium?
Bertrand Competition with Capacity Constraints (cont.)

• We consider symmetric strategies (i.e., both firms use same mixed strategy).
• We use cumulative distribution function $F(.)$ to represent mixed strategy used by either firm.
• What is expected utility of firm 1 when it chooses $p_1$ and firm 2 uses mixed strategy $F(.)$?

$$u_1(p_1, F(.)) = F(p_1) \frac{p_1}{3} + (1 - F(p_1)) \frac{2p_1}{3}$$

• Knowing that each action in support of mixed strategy must yield same utility at equilibrium, we obtain for all $p$ in support of $F(.)$

$$-F(p) \frac{p}{3} + \frac{2p}{3} = k,$$

for some $k \geq 0$, which leads to

$$F(p) = 2 - \frac{3k}{p}$$
Bertrand Competition with Capacity Constraints (cont.)

- Note that upper support of mixed strategy must be at $p = 1$, which implies that $F(1) = 1$

- Combining with preceding, we obtain

$$F(p) = \begin{cases} 
0, & \text{if } 0 \leq p \leq \frac{1}{2} \\
2 - \frac{1}{p}, & \text{if } \frac{1}{2} \leq p \leq 1 \\
1, & \text{if } p \geq 1.
\end{cases}$$
Questions?
Acknowledgement

• This lecture is a slightly modified version of one prepared by
  • Asu Ozdaglar [MIT 6.254]