ECE700.07: Game Theory with Engineering Applications

Lecture 5: Supermodular and Potential Games

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Outline

• Supermodular games
• Potential games
• Congestion games

• Readings:
  • GT Sec. 12.3
Supermodular Games

• Informally, marginal utility of increasing agent’s strategy raises with increases in other agents’ strategies
  • Implication: best response of agent is nondecreasing function of other agents’ strategies

• Why is this interesting?
  • They arise in many models
  • They have nice sensitivity (or comparative statics) properties and behave well under variety of distributed dynamic rules
Partially Ordered Set

• Given a set $S$ and binary relation $\geq$, pair $(S, \geq)$ is partially ordered set if $\geq$ has following properties
  • Reflexive: $x \geq x$ for all $x \in S$
  • Transitive: $x \geq y$ and $y \geq z$ implies that $x \geq z$
  • Antisymmetric: $x \geq y$ and $y \geq x$ implies that $x = y$
Monotonicity of Optimal Solutions

• Consider following problem

\[ x(t) = \arg \max_{x \in X} f(x, t) \]

where \( f: X \times T \to \mathbb{R} \), \( X \subseteq \mathbb{R} \) and \( T \) is some partially ordered set.

• We focus on \( T \subseteq \mathbb{R}^k \) for some \( k \) with usual vector order.
  • For all \( x, y \in T \), \( x \geq y \) if and only if \( x_i \geq y_i \) for all \( i = 1, ..., k \).

• We are interested in conditions under which we can establish that \( x(t) \) is nondecreasing function of \( t \).
Increasing Differences

• Let $X \subseteq \mathbb{R}$ and $T$ be some partially ordered set.
• Function $f : X \times T \to \mathbb{R}$ has increasing differences in $(x, t)$ if for all $x' \geq x$ and $t' \geq t$, we have
  \[ f(x', t') - f(x, t') \geq f(x', t) - f(x, t) \]
• Intuition: Gain to choosing higher $x$ is greater when $t$ is higher.
Increasing Differences (cont.)

• **Lemma:**
  
  • Let $X \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}^k$ for some $k$ be some partially ordered set with usual vector order
  
  • Let $f : X \times T \rightarrow \mathbb{R}$ be a twice continuously differentiable function
  
  • Function $f$ has increasing differences in $x, t$ if and only if
    \[
    \frac{\partial f(x, t')}{\partial x} \geq \frac{\partial f(x, t)}{\partial x}, \quad \forall t' \geq t, x \in X
    \]
    
    Or
    \[
    \frac{\partial^2 f(x, t)}{\partial x \partial t_i} \geq 0, \quad \forall x \in X, t \in T, i = 1, \ldots, k
    \]
Example 1: Network Effects (Positive Externalities)

• Agent $i \in J$ chooses between two products $S_i = \{X_1, X_2\}$
  • E.g., Blu-ray and HD DVD

• Utility of $i$ when subset $J$ of agents use $X_k$ and $i \in J$ is $B_i(J, X_k)$, for $k = 1, 2$

• There exists positive externality if
  \[ B_i(J, X) \leq B_i(J', X), \quad \text{when } J \subseteq J' \]
  i.e., agent $i$ is better off if more users use the same technology as her

• Given $s \in S$, let $X_k(s) = \{i \in J \mid s_i = X_k\}$, for $k = 1, 2$

• Define utility of agent $i$ as
  \[ u_i(s_i, s_{-i}) = \begin{cases} B_i(X_1(s), X_1) & \text{if } s_i = X_1 \\ B_i(X_2(s), X_2) & \text{if } s_i = X_2 \end{cases} \]

• It can be verified that $u_i$ satisfies increasing differences
Example II: Investment Game

• Agents $i \in 1, \ldots, n$ simultaneously choose investment $a_i \in \{0,1\}$

• Utility of agent $i$ is modeled as

$$u_i(a_i, a_{-i}) = \begin{cases} 
\pi \left( \sum_{j=1}^{n} a_j \right) - k & \text{if } a_i = 1 \\
0 & \text{if } a_i = 0 
\end{cases}$$

• If $\pi$ is increasing in aggregate investment, then $u_i$ satisfies increasing differences
Example III: Bertrand Price Competition

- Firm $i \in \{1, \ldots, n\}$ chooses price $p_i \in [0,1]$
- Let $D_i$ denote demand function

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j$$

where $b_i, d_{ij} \geq 0$

- Utility of firm $i$ can be modeled as

$$\pi_i(p_i, p_{-i}) = (p_i - c_i)D_i(p_i, p_{-i})$$

- It can be verified that $\pi$ satisfies increasing differences

$$\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} = d_{ij} \geq 0$$
Example IV: Diamond Search Model

• Agent $i \in \{1, \ldots, n\}$ looks for partner
• Each $i$ chooses search effort $e_i \in [0,1]$
• Cost of search is $c(e_i)$, which is increasing and continuous
• Chances of finding partner increases in $e_i \cdot \sum_{j \neq i} e_j$
• Agent $i$’s utility can be modeled as
  \[ u_i(e_i, e_{-i}) = e_i \cdot \sum_{j \neq i} e_j - c(e_i) \]
• It can be verified that $u_i$ satisfies increasing differences
Monotonicity of Optimal Solutions

• **Theorem (Topkis):**
  
  - Let $X$ be compact set and $T$ be some partially ordered set
  - Assume that $f: X \times T \to \mathbb{R}$ is continuous (or upper semicontinuous) in $x$ for all $t$ and has **increasing differences** in $(x, t)$
  - Define $x(t) = \arg \max_{x \in X} f(x, t)$, then, we have
    - For all $t \in T$, $x(t)$ is **nonempty** and has greatest, $\bar{x}(t)$, and least element, $\underline{x}(t)$
    - For all $t' \geq t$, we have $\bar{x}(t') \geq \bar{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$

• **Summary:** if $f$ has increasing differences, set of optimal solutions $x(t)$ is **nonempty**, and its largest and smallest selections are **non-decreasing**
Supermodular Games

• Strategic game \( \langle I, (S_i), (u_i) \rangle \) is supermodular game if for all \( i \in I \)
  • \( S_i \) is compact subset of \( \mathbb{R} \)
  • \( u_i \) is upper semicontinuous in \( s_i \), continuous in \( s_{-i} \)
  • \( u_i \) has increasing differences in \( (s_i, s_{-i}) \)

• Applying Topkis’s theorem implies that in supermodular game if

\[
B_i(s_{-i}) = \arg \max_{s_i \in S_i} u_i(s_i, s_{-i})
\]

Then:

• \( B_i(s_{-i}) \) has greatest and least element, denoted by \( \bar{B}_i(s_{-i}) \) and \( \underline{B}_i(s_{-i}) \)

• If \( s_{-i}' \geq s_{-i} \), then \( \bar{B}_i(s_{-i}') \geq \bar{B}_i(s_{-i}) \) and \( \underline{B}_i(s_{-i}') \geq \underline{B}_i(s_{-i}) \)
Suppermodular Games (cont.)

• Theorem (Milgrom and Roberts):
  • Let \( (I, (S_i), (u_i)) \) be supermodular game. Then set of strategies that survive iterated strict dominance (ISD) in pure strategies has greatest and least elements coinciding with greatest and least pure strategy Nash Equilibria.

• Corollary:
  • Supermodular games have following properties:
    • Pure strategy NE exist
    • Largest and smallest strategies compatible with ISD, correlated equilibrium and Nash equilibrium are the same
    • If supermodular game has unique NE, it is dominance solvable (and lots of learning and adjustment rules converge to it, e.g., best-response dynamics)
Proof

• Let $S^0 = S$, and $s^0 = (s_1^0, ..., s_l^0)$ be largest element of $S$

• Let $s_i^1 = \overline{B}_i(s_{-i}^0)$

• For all $s_i > s_i^1$ we have

\[ u_i(s_i, s_{-i}) - u(s_i^1, s_{-i}) \leq u_i(s_i, s_{-i}^0) - u(s_i^1, s_{-i}^0) < 0 \]

where first inequality follows by increasing differences, and second one holds because $s_i$ is not best response to $s_{-i}^0$.

• Note that $s_i^1 \leq s_i^0$

• Define $s_i^k = \overline{B}_i(s_{-i}^{k-1})$
Proof (cont.)

• By Corollary (Topkis) we can show if $s_i^k \leq s_i^{k-1}$, then
  $$s_i^{k+1} = \bar{B}_i(s_i^k) \leq \bar{B}_i(s_i^{k-1}) = s_i^k$$

• This shows that \{s_i^k\} is decreasing and bounded from below, and hence it has limit denoted by $\bar{s}_i$
  • Only strategies $s_i \leq \bar{s}_i$ are undominated

• Similarly, we can start with $s^0$ smallest element in $S$ to identify $\underline{s}$

• By construction, for all $i$ and $s_i \in S$, we have
  $$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k)$$

• Taking limit as $k \to \infty$ and using upper semicontinuity of $u_i$ in $s_i$ and its continuity in $s_{-i}$, we obtain
  $$u_i(\bar{s}_i, \bar{s}_{-i}) \geq u_i(s_i, \bar{s}_{-i})$$
  which completes proof by showing that $\bar{s}$ and $\underline{s}$ are NE
Example: Solving Bertrand Game

• Suppose \( n = 2, c_i = 0 \), and demand function is
  \[
  D_i(p_i, p_{-i}) = 1 - 2p_i + p_{-i}
  \]

• Utility function is
  \[
  u_i(p_i, p_{-i}) = p_i (1 - 2p_i + p_{-i})
  \]

• Best response function is
  \[
  B_i(p_{-i}) = \frac{1 + p_{-i}}{4}
  \]
Bertrand Game: Iterated Dominance

• Set $P_i^0 = [0,1]$

• Since $\frac{1}{4} \leq B_i(p_{-i}^0) \leq \frac{1}{2}$, any $p_i < \frac{1}{4}$ or $> \frac{1}{2}$ is strictly dominated

• Therefore, $P_i^1 = [\frac{1}{4}, \frac{1}{2}]$

• Iterating, $P_i^k = [\underline{p}^k, \overline{p}^k]$, where $\underline{p}^k = B_i(\underline{p}^{k-1})$ and $\overline{p}^k = B_i(\overline{p}^{k-1})$

• Calculating these best-responses, we have

  \[
  \overline{p}^k = 1, \frac{2}{4}, \frac{6}{16}, \frac{22}{64}, \frac{86}{256}, \frac{342}{1024}, \ldots \rightarrow \frac{1}{3}
  \]

  \[
  \underline{p}^k = 0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}, \frac{341}{1024}, \ldots \rightarrow \frac{1}{3}
  \]

• Thus, $\left(\frac{1}{3}, \frac{1}{3}\right)$ is unique NE
Potential Games

• Strategic form game is potential game (ordinal potential game, exact potential game) if there exists function $\Phi: S \rightarrow \mathbb{R}$ such that $\Phi(s_i, s_{-i})$ gives information about $u_i(s_i, s_{-i})$ for every $i \in J_0$
  - $\Phi$ is referred to as potential function

• The potential function has natural analogy to "energy" in physics
  - This will be useful both for locating pure strategy Nash equilibria and also for analysis of "myopic" dynamics
Potential Functions and Games

- Function $\Phi: S \rightarrow \mathbb{R}$ is called **ordinal potential function** for strategic form game $G = (I, (S_i), (u_i))$ if for all $i \in I$ and all $s_{-i} \in S_{-i}$
  
  $$u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0 \iff \Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0, \forall x, z \in S_i$$

- $G$ is called **ordinal potential game** if it admits ordinal potential

- Function $\Phi: S \rightarrow \mathbb{R}$ is called **(exact) potential function** for strategic form game $G$ if for all $i \in I$ and all $s_{-i} \in S_{-i}$
  
  $$u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i}), \forall x, z \in S_i$$

- $G$ is called **(exact) potential game** if it admits (exact) potential
Potential Function

• Potential function assigns real value for every $s \in S$

• When we represent agent utilities with matrix (in finite games), we can also represent potential function as matrix, each entry corresponding to vector of strategies

• Example: Prisoner’s dilemma

$$G = \begin{pmatrix} (-1, -1) & (-3, 0) \\ (0, -3) & (-2, -2) \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}$$
Pure Strategy Nash Equilibria in Ordinal Potential Games

• **Theorem:**

  • Every finite ordinal potential game has at least one pure strategy Nash equilibrium

• **Proof:**

  • Global maximum of $\Phi$ is pure strategy Nash equilibrium

    $$u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) \geq 0 \iff \Phi(s_i^*, s_{-i}^*) - \Phi(s_i, s_{-i}^*) \geq 0$$

• Note, however, that there may also be other pure strategy Nash equilibria corresponding to **local maxima**
Example 1: Cournot Competition

• Firms choose quantity $q_i \in (0, \infty)$
• Utility function for firm $i$ is given by $u_i(q_i, q_{-i}) = q_i(P(Q) - c)$
• We define function $\Phi(q) = (\prod_{i=1}^{n} q_i)(P(Q) - c)$
• Note that for all $i$ and all $q_{-i} > 0$,
  $$u_i(q_i, q_{-i}) - u_i(q'_i, q_{-i}) > 0 \iff \Phi(q_i, q_{-i}) - \Phi(q'_i, q_{-i}) > 0, \forall q_i, q'_i > 0$$
• $\Phi$ is, therefore, ordinal potential function for this game
Example II: Cournot Competition (again!)

• Suppose that \( P(Q) = a - bQ \) and costs \( c_i(q_i) \) are arbitrary

• Define potential function as

\[
\Phi^*(q_1, \ldots, q_n) = a \sum_{i=1}^{n} q_i - b \sum_{i=1}^{n} \sum_{j=i}^{n} q_i q_j - \sum_{i=1}^{n} c_i(q_i)
\]

• It can be shown that for all \( i \) and all \( q_{-i} \),
  • \( u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi^*(q_i, q_{-i}) - \Phi^*(q_i', q_{-i}), \forall q_i, q_i' > 0 \)

• \( \Phi \) is exact potential function for this game
Simple Dynamics in Finite Ordinal Potential Games

• **Path** in strategy space $S$ is sequence of strategy vectors $(s^0, s^1, ...)$ such that every two consecutive strategies differ in one coordinate (*i.e.*, exactly in one agent’s strategy)

• **Improvement path** is path $(s^0, s^1, ...)$ such that,
  $$u_{i_k}(s^k) < u_{i_k}(s^{k+1})$$
  where $s^k$ and $s^{k+1}$ differ in $i_k^{th}$ coordinate
  
  • Utility improves for agent who changes her strategy

• Improvement path can be thought of as generated dynamically by “myopic agents”, who update their strategies according to 1-sided better reply dynamic
Simple Dynamics in Finite Ordinal Potential Games (cont.)

• **Proposition:**
  - In every finite ordinal potential game, every improvement path is finite

• **Proof:**
  - If \((s^0, s^1, \ldots)\) is improvement path, then we have
    \[ \Phi(s^0) < \Phi(s^1) < \ldots \]
  - Since strategy space is finite, \(\Phi\) takes finitely many values and sequence must end in finitely many steps

• **Implication:**
  - In finite ordinal potential games, every “maximal” improvement path must terminate in equilibrium point
  - Simple myopic learning process based on 1-sided better reply dynamic converges to equilibrium set
  - We will see that other natural simple dynamics also converge to pure equilibrium for potential games
Characterization of Finite Exact Potential Games

• For finite path $\gamma = (s^0, ..., s^N)$, let

$$I(\gamma) = \sum_{i=1}^{N} u_{m_i}(s^i) - u_{m_i}(s^{i-1})$$

where $m_i$ denotes agent changing her strategy in $i^{th}$ step

• Path $\gamma = (s^0, ..., s^N)$ is closed if $s^0 = s^N$, and it is simple closed path if in addition to being closed, $s^l \neq s^k$ for every $0 \leq l \neq k \leq N - 1$

• **Theorem:**
  • Game $G$ is exact potential game if and only if for all finite simple closed paths, $\gamma$, $I(\gamma) = 0$
  • It is sufficient to check simple closed paths of length 4
Infinite Potential Games

• **Proposition:**
  • If \( G \) is continuous potential game with compact strategy sets, then \( G \) has at least one pure strategy Nash equilibrium

• **Proposition:**
  • If \( G \) is game such that \( S_i \subseteq \mathbb{R} \) and utility functions \( u_i: S \to \mathbb{R} \) are continuously differentiable, then \( \Phi: S \to \mathbb{R} \) is potential for \( G \) if and only if \( \Phi \) is continuously differentiable and

\[
\frac{\partial u_i(s)}{\partial s_i} = \frac{\partial \Phi(s)}{\partial s_i}
\]

for all \( i \in I \) and all \( s \in S \)
Congestion Games

- **Congestion model**: \( C = \langle J, \mathcal{M}, (S_i)_{i \in J}, (c^j)_{j \in \mathcal{M}} \rangle \) where:
  - \( J = \{1, ..., n\} \) is set of agents
  - \( \mathcal{M} = \{1, ..., m\} \) is set of resources
  - \( S_i \) is set of resource combinations (e.g., links or common resources) that agent \( i \) can use
    - \( s_i \in S_i \) is agent \( i \)'s strategy corresponding to subset of resources that she uses
  - \( c^j(k) \) is **negative of cost** to each agent who uses resource \( j \) if \( k \) agents use it
  - Define congestion game \( \langle J, (S_i), (u_i) \rangle \) with utilities

\[
  u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j)
\]

where \( k_j \) is number of users of resource \( j \) under strategy \( s \)
Congestion and Potential Games

• **Theorem (Rosenthal (73)):**
  - Every congestion game is potential game and thus has pure strategy Nash equilibrium

• **Proof:**
  - For each $j$ define $\bar{k}^i_j$ as usage of resource $j$ excluding agent $i$
    
    $$
    \bar{k}^i_j = \sum_{i' \neq i} I[j \in s_{i'}]
    $$

    where $I[j \in s_{i'}]$ is indicator for event that $j \in s_{i'}$.

  - With this notation, utility difference of agent $i$ from two strategies $s_i$ and $s_i'$ (when others are using strategy profile $s_{-i}$) is
    
    $$
    u_i(s_i,s_{-i}) - u_i(s'_i,s_{-i}) = \sum_{j \in s_i} c^j (\bar{k}^i_j + 1) - \sum_{j \in s'_i} c^j (\bar{k}^i_j + 1)
    $$
Proof (cont.)

• Now consider function

$$\Phi(s) = \sum_{j \in \cup s_i} \sum_{k=1}^{k_j} c^j(k)$$

• We can also write

$$\Phi(s_i, s_{-i}) = \sum_{j \in \cup_{l \neq i} s_l} \sum_{k=1}^{\tilde{k}_j} c^j(k) + \sum_{j \in s_l} c^j(\tilde{k}_j + 1)$$

• Therefore, we have

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = \sum_{j \in s_l} c^j(\tilde{k}_j + 1) - \sum_{j \in s_l'} c^j(\tilde{k}_j + 1) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})$$
Questions?
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