ECE700.07: Game Theory with Engineering Applications

Lecture 5: Supermodular and Potential Games

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Outline

• Supermodular games
• Potential games
• Congestion games

• Readings:
  • GT Sec. 12.3
Supermodular Games

• Informally, **marginal utility** of increasing agent's strategy raises with increases in other agents’ strategies
  
  • Implication: best response of agent is **nondecreasing function** of other agents’ strategies

• Why is this interesting?
  
  • They **arise in many models**
  
  • They have nice sensitivity (or comparative statics) properties and **behave well** under variety of distributed dynamic rules
Partially Ordered Set

• Given a set $S$ and binary relation $\geq$, pair $(S, \geq)$ is partially ordered set if $\geq$ has following properties
  
  • Reflexive: $x \geq x$ for all $x \in S$
  
  • Transitive: $x \geq y$ and $y \geq z$ implies that $x \geq z$
  
  • Antisymmetric: $x \geq y$ and $y \geq x$ implies that $x = y$
Monotonicity of Optimal Solutions

• Consider following problem

\[ x(t) = \arg \max_{x \in X} f(x, t) \]

where \( f: X \times T \to \mathbb{R} \), \( X \subseteq \mathbb{R} \) and \( T \) is some partially ordered set

• We focus on \( T \subseteq \mathbb{R}^k \) for some \( k \) with usual vector order

  • For all \( x, y \in T \), \( x \geq y \) if and only if \( x_i \geq y_i \) for all \( I = 1, \ldots, k \)

• We are interested in conditions under which we can establish that \( x(t) \) is nondecreasing function of \( t \)
Increasing Differences

• Let $X \subseteq \mathbb{R}$ and $T$ be some partially ordered set

• Function $f : X \times T \to \mathbb{R}$ has **increasing differences** in $(x, t)$ if for all $x' \geq x$ and $t' \geq t$, we have

\[
  f(x', t') - f(x, t') \geq f(x', t) - f(x, t)
\]

• **Intuition:** Gain to choosing higher $x$ is greater when $t$ is higher
Increasing Differences (cont.)

• **Lemma:**
  - Let $X \subset \mathbb{R}$ and $T \subset \mathbb{R}^k$ for some $k$ be some partially ordered set with usual vector order.
  - Let $f: X \times T \rightarrow \mathbb{R}$ be twice continuously differentiable function.
  - Function $f$ has increasing differences in $x, t$ if and only if
    \[
    \frac{\partial f(x, t')}{\partial x} \geq \frac{\partial f(x, t)}{\partial x}, \quad \forall t' \geq t, x \in X
    \]
    Or
    \[
    \frac{\partial^2 f(x, t)}{\partial x \partial t_i} \geq 0, \quad \forall x \in X, t \in T, i = 1, \ldots, k
    \]

Example 1: Network Effects (Positive Externalities)

- Agent \( i \in J \) chooses between two products \( S_i = \{X_1, X_2\} \)
  - \( E.g. \), Blu-ray and HD DVD
- Utility of \( i \) when subset \( J \) of agents use \( X_k \) and \( i \in J \) is \( B_i(J, X_k) \), for \( k = 1, 2 \)
- There exists positive externality if
  \[
  B_i(J, X) \leq B_i(J', X), \quad \text{when } J \subseteq J'
  \]
  i.e., agent \( i \) is better off if more users use the same technology as her
- Given \( s \in S \), let \( X_k(s) = \{i \in J \mid s_i = X_k\} \), for \( k = 1, 2 \)
- Define utility of agent \( i \) as
  \[
  u_i(s_i, s_{-i}) = \begin{cases} 
  B_i(X_1(s), X_1) & \text{if } s_i = X_1 \\
  B_i(X_2(s), X_2) & \text{if } s_i = X_2 
  \end{cases}
  \]
- It can be verified that \( u_i \) satisfies increasing differences
Example II: Investment Game

• Agents $i \in 1, \ldots, n$ simultaneously choose investment $a_i \in \{0,1\}$

• Utility of agent $i$ is modeled as

$$u_i(a_i, a_{-i}) = \begin{cases} 
\pi \left( \sum_{j=1}^{n} a_j \right) - k & \text{if } a_i = 1 \\
0 & \text{if } a_i = 0
\end{cases}$$

• If $\pi$ is increasing in aggregate investment, then $u_i$ satisfies increasing differences
Example III: Bertrand Price Competition

• Firm \( i \in \{1, ..., n\} \) chooses price \( p_i \in [0,1] \)

• Let \( D_i \) denote demand function

\[
D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j
\]

where \( b_i, d_{ij} \geq 0 \)

• Utility of firm \( i \) can be modeled as

\[
\pi_i(p_i, p_{-i}) = (p_i - c_i)D_i(p_i, p_{-i})
\]

• It can be verified that \( \pi \) satisfies increasing differences

\[
\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} = d_{ij} \geq 0
\]
Example IV: Diamond Search Model

- Agent $i \in \{1, \ldots, n\}$ looks for partner
- Each $i$ chooses search effort $e_i \in [0,1]$
- Cost of search is $c(e_i)$, which is increasing and continuous
- Chances of finding partner increases in $e_i \cdot \sum_{j \neq i} e_j$
- Agent $i$’s utility can be modeled as
  \[ u_i(e_i, e_{-i}) = e_i \cdot \sum_{j \neq i} e_j - c(e_i) \]
- It can be verified that $u_i$ satisfies increasing differences
Monotonicity of Optimal Solutions

• **Theorem (Topkis):**
  
  - Let $X$ be compact set and $T$ be some partially ordered set
  - Assume that $f: X \times T \to \mathbb{R}$ is continuous (or upper semicontinuous) in $x$ for all $t$ and has **increasing differences** in $(x, t)$
  - Define $x(t) = \arg \max_{x \in X} f(x, t)$, then, we have
    - For all $t \in T$, $x(t)$ is **nonempty** and has greatest, $\bar{x}(t)$, and least element, $\underline{x}(t)$
    - For all $t' \geq t$, we have $\bar{x}(t') \geq \bar{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$

• **Summary:** if $f$ has increasing differences, set of optimal solutions $x(t)$ is **nonempty**, and its largest and smallest selections are non-decreasing
Supermodular Games

• Strategic game \( \langle \mathcal{I}, (S_i), (u_i) \rangle \) is supermodular game if for all \( i \in \mathcal{I} \)
  • \( S_i \) is compact subset of \( \mathbb{R} \)
  • \( u_i \) is upper semicontinuous in \( s_i \), continuous in \( s_{-i} \)
  • \( u_i \) has increasing differences in \( (s_i, s_{-i}) \)

• Applying Topkis’s theorem implies that in supermodular game if

\[
B_i(s_{-i}) = \arg \max_{s_i \in S_i} u_i(s_i, s_{-i})
\]

Then:

• \( B_i(s_{-i}) \) has greatest and least element, denoted by \( \overline{B}_i(s_{-i}) \) and \( B_i(s_{-i}) \)
• If \( s'_{-i} \geq s_{-i} \), then \( \overline{B}_i(s'_{-i}) \geq \overline{B}_i(s_{-i}) \) and \( B_i(s'_{-i}) \geq B_i(s_{-i}) \)
Suppermodular Games (cont.)

• Theorem (Milgrom and Roberts):
  • Let $\langle J, (S_i), (u_i) \rangle$ be supermodular game. Then set of strategies that survive iterated strict dominance (ISD) in pure strategies has greatest and least elements coinciding with greatest and least pure strategy Nash Equilibria.

• Corollary:
  • Supermodular games have following properties:
    • Pure strategy NE exist
    • Largest and smallest strategies compatible with ISD, correlated equilibrium and Nash equilibrium are the same
    • If supermodular game has unique NE, it is dominance solvable (and lots of learning and adjustment rules converge to it, e.g., best-response dynamics)
Proof

• Let $S^0 = S$, and $s^0 = (s^0_1, ..., s^0_i)$ be largest element of $S$

• Let $s^1_i = \overline{B}_i(s^0_{-i})$

• For all $s_i > s^1_i$ we have

$$u_i(s_i, s_{-i}) - u(s^1_i, s_{-i}) \leq u_i(s_i, s^0_{-i}) - u(s^1_i, s^0_{-i}) < 0$$

where first inequality follows by increasing differences, and second one holds because $s_i$ is not best response to $s^0_{-i}$

• Note that $s^1_i \leq s^0_i$

• Define $s^k_i = \overline{B}_i(s^{k-1}_{-i})$
Proof (cont.)

- By Corollary (Topkis) we can show if $s_i^k \leq s_i^{k-1}$, then
  
  $$s_i^{k+1} = \overline{B}_i(s_i^k) \leq \overline{B}_i(s_i^{k-1}) = s_i^k$$

- This shows that $\{s_i^k\}$ is decreasing and bounded from below, and hence it has limit denoted by $\overline{s}_i$
  - Only strategies $s_i \leq \overline{s}_i$ are undominated

- Similarly, we can start with $s^0$ smallest element in $S$ to identify $\underline{s}$

- By construction, for all $i$ and $s_i \in S_i$, we have
  
  $$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i^k, s_{-i}^k)$$

- Taking limit as $k \to \infty$ and using upper semicontinuity of $u_i$ in $s_i$ and its continuity in $s_{-i}$, we obtain
  
  $$u_i(\overline{s}_i, \overline{s}_{-i}) \geq u_i(s_i, \overline{s}_{-i})$$

  which completes proof by showing that $\overline{s}$ and $\underline{s}$ are NE
Example: Solving Bertrand Game

• Suppose $n = 2$, $c_i = 0$, and demand function is
  \[ D_i(p_i, p_{-i}) = 1 - 2p_i + p_{-i} \]

• Utility function is
  \[ u_i(p_i, p_{-i}) = p_i(1 - 2p_i + p_{-i}) \]

• Best response function is
  \[ B_i(p_{-i}) = \frac{1 + p_{-i}}{4} \]
Bertrand Game: Iterated Dominance

- Set $P_i^0 = [0,1]$

- Since $\frac{1}{4} \leq B_i(p_{-i}) \leq \frac{1}{2}$, any $p_i < \frac{1}{4}$ or $> \frac{1}{2}$ is strictly dominated

- Therefore, $P_i^1 = [\frac{1}{4}, \frac{1}{2}]$

- Iterating, $P_i^k = [p_i^k, \bar{p}_i^k]$, where $p_i^k = B_i(p_{-i}^{k-1})$ and $\bar{p}_i^k = B_i(\bar{p}_{-i}^{k-1})$

- Calculating these best-responses, we have

  $\bar{p}_i^k = 1, 2, 6, 22, 86, 342, \ldots \rightarrow \frac{1}{3}$

  $p_i^k = 0, 1, 5, 21, 85, 341, \ldots \rightarrow \frac{1}{3}$

- Thus, $(\frac{1}{3}, \frac{1}{3})$ is unique NE
Potential Games

- Strategic form game is **potential game** (ordinal potential game, exact potential game) if there exists function $\Phi: S \to \mathbb{R}$ such that $\Phi(s_i, s_{-i})$ gives information about $u_i(s_i, s_{-i})$ for every $i \in I_0$
  - $\Phi$ is referred to as **potential function**

- The potential function has natural analogy to “**energy**” in physics
  - This will be useful both for locating pure strategy Nash equilibria and also for analysis of “**myopic**” dynamics
Potential Functions and Games

• Function $\Phi: S \rightarrow \mathbb{R}$ is called ordinal potential function for strategic form game $G = (I, (S_i), (u_i))$ if for all $i \in I$ and all $s_{-i} \in S_{-i}$
  
  $u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0 \iff \Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0 \, , \forall x, z \in S_i$

• $G$ is called ordinal potential game if it admits ordinal potential

• Function $\Phi: S \rightarrow \mathbb{R}$ is called (exact) potential function for strategic form game $G$ if for all $i \in I$ and all $s_{-i} \in S_{-i}$
  
  $u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i}) \, , \forall x, z \in S_i$

• $G$ is called (exact) potential game if it admits (exact) potential
Potential Function

- Potential function assigns real value for every $s \in S$

- When we represent agent utilities with matrix (in finite games), we can also represent potential function as matrix, each entry corresponding to vector of strategies

- Example: Prisoner’s dilemma

$$G = \begin{pmatrix} (-1, -1) & (-3, 0) \\ (0, -3) & (-2, -2) \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}$$
Pure Strategy Nash Equilibria in Ordinal Potential Games

• **Theorem:**
  
  • Every finite ordinal potential game has at least one pure strategy Nash equilibrium

• **Proof:**
  
  • Global maximum of $\Phi$ is pure strategy Nash equilibrium
    
    $$u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) \geq 0 \iff \Phi(s_i^*, s_{-i}^*) - \Phi(s_i, s_{-i}^*) \geq 0$$

  • Note, however, that there may also be other pure strategy Nash equilibria corresponding to local maxima
Example I: Cournot Competition

• Firms choose quantity $q_i \in (0, \infty)$

• Utility function for firm $i$ is given by $u_i(q_i, q_{-i}) = q_i(P(Q) - c)$

• We define function $\Phi(q) = (\prod_{i=1}^{n} q_i)(P(Q) - c)$

• Note that for all $i$ and all $q_{-i} > 0$,
  
  $u_i(q_i, q_{-i}) - u_i(q'_i, q_{-i}) > 0 \iff \Phi(q_i, q_{-i}) - \Phi(q'_i, q_{-i}) > 0, \forall q_i, q'_i > 0$

• $\Phi$ is, therefore, ordinal potential function for this game
Example II: Cournot Competition (again!)

• Suppose that $P(Q) = a - bQ$ and costs $c_i(q_i)$ are arbitrary

• Define potential function as

$$
\Phi^*(q_1, ..., q_n) = a \sum_{i=1}^{n} q_i - b \sum_{i=1}^{n} \sum_{j=i}^{n} q_i q_j - \sum_{i=1}^{n} c_i(q_i)
$$

• It can be shown that for all $i$ and all $q_{-i}$,

  • $u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi^*(q_i, q_{-i}) - \Phi^*(q_i', q_{-i}), \forall q_i, q_i' > 0$

  • $\Phi$ is exact potential function for this game
Simple Dynamics in Finite Ordinal Potential Games

• **Path** in strategy space $S$ is sequence of strategy vectors $(s^0, s^1, ...)$ such that every two consecutive strategies differ in one coordinate (i.e., exactly in one agent’s strategy)

• **Improvement path** is path $(s^0, s^1, ...)$ such that,
  
  \[ u_{i_k}(s^k) < u_{i_k}(s^{k+1}) \]

  where $s^k$ and $s^{k+1}$ differ in $i_k^{th}$ coordinate
  
  • Utility improves for agent who changes her strategy

• Improvement path can be thought of as generated dynamically by “myopic agents”, who update their strategies according to 1-sided better reply dynamic
Simple Dynamics in Finite Ordinal Potential Games (cont.)

• **Proposition:**
  - In every finite ordinal potential game, every improvement path is finite

• **Proof:**
  - If \((s^0, s^1, \ldots)\) is improvement path, then we have
    \[
    \Phi(s^0) < \Phi(s^1) < \ldots
    \]
  - Since strategy space is finite, \(\Phi\) takes finitely many values and sequence must end in finitely many steps

• **Implication:**
  - In finite ordinal potential games, every “maximal” improvement path must terminate in equilibrium point
  - Simple myopic learning process based on 1-sided better reply dynamic converges to equilibrium set
  - We will see that other natural simple dynamics also converge to pure equilibrium for potential games
Characterization of Finite Exact Potential Games

• For finite path $\gamma = (s^0, ..., s^N)$, let

$$I(\gamma) = \sum_{i=1}^{N} u_{m_i}(s^i) - u_{m_i}(s^{i-1})$$

where $m_i$ denotes agent changing her strategy in $i^{th}$ step

• Path $\gamma = (s^0, ..., s^N)$ is closed if $s^0 = s^N$, and it is simple closed path if in addition to being closed, $s^l \neq s^k$ for every $0 \leq l \neq k \leq N - 1$

• Theorem:

  • Game $G$ is exact potential game if and only if for all finite simple closed paths, $\gamma$, $I(\gamma) = 0$
  
  • It is sufficient to check simple closed paths of length 4
Infinite Potential Games

• **Proposition:**
  • If $G$ is continuous potential game with compact strategy sets, then $G$ has at least one pure strategy Nash equilibrium

• **Proposition:**
  • If $G$ is game such that $S_i \subseteq \mathbb{R}$ and utility functions $u_i: S \rightarrow \mathbb{R}$ are continuously differentiable, then $\Phi: S \rightarrow \mathbb{R}$ is potential for $G$ if and only if $\Phi$ is continuously differentiable and

  \[ \frac{\partial u_i(s)}{\partial s_i} = \frac{\partial \Phi(s)}{\partial s_i} \]

  for all $i \in I$ and all $s \in S$
Congestion Games

- **Congestion model**: $C = \{I, M, (S_i)_{i \in I}, (c^j)_{j \in M}\}$ where:
  - $I = \{1, \ldots, n\}$ is the set of agents
  - $M = \{1, \ldots, m\}$ is the set of resources
  - $S_i$ is the set of resource combinations (e.g., links or common resources) that agent $i$ can use
    - $s_i \in S_i$ is agent $i$’s strategy corresponding to subset of resources that she uses
  - $c^j(k)$ is the negative of cost to each agent who uses resource $j$ if $k$ agents use it
  - Define congestion game $\langle I, (S_i), (u_i) \rangle$ with utilities
    - $u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j)$
      - where $k_j$ is the number of users of resource $j$ under strategy $s$
Congestion and Potential Games

• **Theorem (Rosenthal (73))**: Every congestion game is potential game and thus has pure strategy Nash equilibrium

• **Proof**:
  • For each $j$ define $\overline{k}_j^i$ as usage of resource $j$ excluding agent $i$
    
    $$\overline{k}_j^i = \sum_{i' \neq i} I[j \in s_{i'}]$$

    where $I[j \in s_{i'}]$ is indicator for event that $j \in s_{i'}$
  • With this notation, utility difference of agent $i$ from two strategies $s_i$ and $s'_i$ (when others are using strategy profile $s_{-i}$) is
    
    $$u_i(s_i,s_{-i}) - u_i(s'_i,s_{-i}) = \sum_{j \in s_i} c^j (\overline{k}_j^i + 1) - \sum_{j \in s'_i} c^j (\overline{k}_j^i + 1)$$
Proof (cont.)

• Now consider function

\[ \Phi(s) = \sum_{j \in \bigcup s_i} \sum_{k=1}^{k_j} c^j(k) \]

• We can also write

\[ \Phi(s_i, s_{-i}) = \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \sum_{k=1}^{\bar{k}_j^i} c^j(k) + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) \]

• Therefore, we have

\[ \Phi(s_i, s_{-i}) - \Phi(s'_i, s_{-i}) = \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s'_i} c^j(\bar{k}_j^i + 1) = u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) \]
Questions?
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