ECE700.07: Game Theory with Engineering Applications

Lecture 5: Extensive Form Games

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Outline

• Extensive Form Games with Perfect Information
• Backward Induction and Subgame Perfect Nash Equilibrium
• One-stage Deviation Principle
• Applications

• Readings:
  • GT Chapter 3 (skim through Sections 3.4 and 3.6), and Sections 4.1-4.2
Extensive Form Games

- So far, we have studied strategic form games
  - Model one-shot games
  - Agents choose their actions once and for all simultaneously
- In this lecture, we will study extensive form games
  - Model multi-agent sequential decision making
- Our focus will be on multi-stage games with observed actions where:
  - All previous actions are observed, i.e., each agent is perfectly informed of all previous events
  - Some agents may move simultaneously at some stage $k$
- Extensive form games can be conveniently represented by game trees
- Additional component of model is histories (i.e., sequences of action profiles)
Example I: Entry Deterrence Game

- There are two agents
- Agent 1, entrant chooses to enter market or stay out
- Agent 2, incumbent, after observing action of entrant, chooses to accommodate her or fight with her
- Utility for each of action profiles (or histories) are given by pair \((x, y)\) at leaves of game tree
  - \(x\) denotes utility of agent 1 (entrant) and \(y\) denotes utility of agent 2 (incumbent)
Example II: Investment in Duopoly

- There are two agents in market
- Agent 1 chooses to invest or not invest
- After agent 1 chooses her action, both agents engage in Cournot competition
  - If agent 1 invests, then they will engage in Cournot game with \( c_1 = 0 \) and \( c_2 = 2 \)
  - Otherwise, they will engage in Cournot game with \( c_1 = c_2 = 2 \)
- We can also assume that there is fixed cost of \( C \) for agent 1 to invest
Extensive Form Game Model

- Set of agents, $I = \{1, \ldots, I\}$
- Histories: set $H$ of sequences which can be finite or infinite
  - $h^0 = \emptyset$ initial history
  - $a^0 = (a^0_1, \ldots, a^0_I)$ stage 0 action profile
  - $h^1 = a^0$ history after stage 0
  - \vdots
  - $h^{k+1} = (a^0, \ldots, a^k)$ history after stage $k$
Extensive Form Game Model (cont.)

- If number of stages is finite, then game is **finite horizon game**

- Let \( H^k = \{h^k\} \) be set of all possible stage \( k \) histories
  - If game has \( k \) stages, \( H^k \) is set of all possible **terminal histories**

- Pure strategies for agent \( i \) is defined as contingency plan for every possible history \( h^k \)
  - Let \( S_i(H^k) = \bigcup_{h^k \in H^k} S_i(h^k) \) be set of actions available to agent \( i \) at stage \( k \)
  - Let \( s_i^k: H^k \to S_i(H^k) \) such that \( s_i^k(h^k) \in S_i(h^k) \)
  - Pure strategy of agent \( i \) is set of sequences \( s_i = \{s_i^k\}_{k=0}^K \)
    - Collection of maps from all possible histories into available actions
    - Observe that path of strategy profile \( s \) is \( a^0 = s^0(h^0), a^1 = s^1(a^0), a^2 = s^2(a^0, a^1), \ldots \)

- Preferences are defined on outcomes of game, \( u_i: H^k \to \mathbb{R} \)
Strategies in Extensive Form Games

- Agent 1’s strategies: \( s^1 : H^0 = \emptyset \rightarrow S_1 = \{C, D\} \)
  - Two possible strategies: \( C \) and \( D \)
- Agent 2’s strategies: \( s^2 : H^1 = \{\{C\}, \{D\}\} \rightarrow S_2 \)
  - Four possible strategies: \( EG, EH, FG \) and \( FH \)
- If \( s = (C, EG) \), then outcome is \( \{C, E\} \)
Example: Sequential Matching Pennies

• Consider following two-stage extensive form version of matching pennies

• How many strategies does agent 2 have?
Sequential Matching Pennies (cont.)

• Recall: strategy should be complete contingency plan

• Therefore: Agent 2 has four strategies:
  • Heads following heads, heads following tails $HH$
  • Heads following heads, tails following tails $HT$
  • Tails following heads, tails following tails $TT$
  • Tails following heads, heads following tails $TH$
Sequential Matching Pennies (cont.)

• From extensive form we can go to strategic form representation

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>HH</th>
<th>HT</th>
<th>TT</th>
<th>TH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>(-1, 1)</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>Tails</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
</tr>
</tbody>
</table>

• What will happen in this game?
Sequential Matching Pennies (cont.)

- Can we go from strategic form to extensive form representation as well?
  
- These extensive form games are representations of simultaneous-move matching pennies

- Loops represent information sets of agents who move at that stage
  - These are imperfect information games

- These games represent exactly the same strategic situation: each agent chooses her action not knowing choice of her opponent

- Note: For consistency, first number is still agent 1’s utility
Information Sets

- Information sets model information agents have when they are choosing their actions.
- They can be viewed as generalization of history.
- Information sets, $h \in H$, partition nodes of game tree:
  - If set $h(x)$ contains node $x$, it means that agent who is choosing action at $x$ is uncertain if she is at $x$ or at some other $x' \in h(x)$.
- We require that if $x' \in h(x)$, then same agent moves at $x$ and $x'$ and also that actions available at $x$ and $x'$ are the same, $A(x) = A(x')$.
- Game has perfect information if all its information sets are singletons:
  - i.e., all nodes are in their own information set.
Entry Deterrence Game

• What is equivalent strategic form representation?

<table>
<thead>
<tr>
<th></th>
<th>Entrant A</th>
<th>Entrant F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incumbent In</td>
<td>(2, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Incumbent Out</td>
<td>(1, 2)</td>
<td>(1, 2)</td>
</tr>
</tbody>
</table>

• Two pure Nash equilibrium: (In, A) and (Out, F)
Are These Equilibria Reasonable?

- (Out, F) is sustained by **noncredible threat** of monopolist
- Equilibrium notion for extensive form games
  - **Subgame Perfect (Nash) Equilibrium**
  - It requires each agent’s strategy to be “**optimal**” not only at start of game, but also after every history
- For finite horizon games, found by **backward induction**
- For infinite horizon games, characterization in terms of **one-stage deviation principle**
Subgames

• To define subgame perfection formally, we first define subgames
  • Informally, subgame is portion of game that can be analyzed as game in its own right
• Recall that game \( G \) is represented by game tree
• Denote set of nodes of \( G \) by \( V_G \)
• Recall that history \( h^k \) denotes strategy profile of game after \( k \) stages
• In perfect information game, each node \( x \in V_G \) corresponds to unique history \( h^k \) and vice versa
  • This is not necessarily true in imperfect information games
Subgames (cont.)

- Subgame $G'$ of extensive form game $G$ consists of single node and all its successors in $G$, with the property that if $x' \in V_{G'}$ and $x'' \in h(x')$, then $x'' \in V_{G'}$
  - Information sets and utilities of subgame are inherited from original game
- This definition requires that all successors of node is in subgame and that subgame does not “chop up” any information set
  - This ensures that subgame can be analyzed in its own right
  - This implies that subgame starts with node $x$ with singleton information set, i.e., $h(x) = x$
- In perfect information games, subgames coincide with nodes or stage $k$ histories $h^k$
  - In this case, we use notation $h^k$ or $G(h^k)$ to denote subgame
- Restriction of strategy $s$ to subgame $G'$, $s_{G'}$, is profile implied by $s$ in subgame $G'$
Subgames: Examples

- Recall two-stage extensive-form version of matching pennies
- In this game, there are two proper subgames and game itself which is also a subgame, and thus total of three subgames
Subgame Perfect Equilibrium

• Strategy profile $s^*$ is **Subgame Perfect Nash equilibrium (SPE)** in game $G$ if for any subgame $G'$ of $G$, $s^*_{G'}$ is Nash equilibrium of $G'$

• Loosely speaking, subgame perfection will remove noncredible threats, since these will not be Nash equilibria in appropriate subgames

• In entry deterrence game, following entry, $F$ is not best response, and thus not a Nash equilibrium of corresponding subgame.
  • Therefore, (Out, F) is not SPE.

• How to find SPE?
  • One could find all of Nash equilibria, for example as in entry deterrence game, then eliminate those that are not subgame perfect
  • But there are more economical ways of doing it
Backward Induction

- Backward induction refers to starting from last subgames of finite game, then finding best response strategy profiles or Nash equilibria in subgames, then assigning these strategies profiles and associated payoffs to be subgames, and moving successively towards beginning of game.
Backward Induction (cont.)

• **Theorem:**
  - Backward induction gives entire set of SPE

• **Proof:**
  - Backward induction makes sure that in restriction of strategy profile in question to any subgame is Nash equilibrium

• Backward induction is straightforward for games with perfect information and finite horizon

• For imperfect information games, backward induction proceeds similarly
  - We identify subgames starting from leaves of game tree and replace it with one of Nash equilibrium utilities in subgame

• For infinite horizon games: we will rely on useful characterization of subgame perfect equilibria given by “one stage deviation principle”
Existence of Subgame Perfect Equilibria

• **Theorem:**
  • Every finite perfect information extensive form game has pure strategy SPE

• **Proof:**
  • Start from end by backward induction and at each step (at least) one strategy is best response

• **Theorem:**
  • Every finite extensive form game G has SPE

• **Proof:**
  • Same argument as previous theorem, except that some subgames need not have perfect information and may have mixed strategy equilibria
One-stage Deviation Principle

- Focus on multi-stage games with observed actions (perfect information games)
- One-stage deviation principle is essentially **principle of optimality** of dynamic programming
- We first state it for finite horizon games
- **Theorem (One-stage deviation principle):**
  - For finite horizon multi-stage games with observed actions, $s^*$ is subgame perfect equilibrium if and only if for all $i$, $t$ and $h^t$, we have
    $$ u_i(s^*_i, s^*_{-i} | h^t) \geq u_i(s_i, s^*_{-i} | h^t) $$
    for all $s_i$ satisfying
    $$ s_i(h^t) \neq s^*_i(h^t), $$
    $$ s_{i|h^t}(h^{t+k}) = s^*_{i|h^t}(h^{t+k}), \text{ for all } k > 0, \text{ and all } h^{t+k} \in \mathcal{G}(h^t) $$
  - Informally, $s$ is SPE if and only if no agent $i$ can gain by deviating from $s$ in single stage and conforming to $s$ thereafter
One-stage Deviation Principle for Infinite Horizon Games

• Proof of one-stage deviation principle for finite horizon games relies on idea that if strategy satisfies one stage deviation principle then that strategy cannot be improved upon by finite number of deviations

• This leaves open possibility that agents may gain by infinite sequence of deviations, which we exclude using the following condition

• Definition:
  • Consider infinite horizon extensive form game \( G^\infty \)
  • Let \( h \) denote an \( \infty \)-horizon history, \( i.e., \ h = (a^0, a^1, a^2, ...) \), is infinite sequence of actions
  • Let \( h^t = (a^0, ..., a^{t-1}) \) be restriction to first \( t \) stages
  • Game \( G^\infty \) is continuous at infinity if for all agent \( i \), utility \( u_i \) satisfies
    \[
    \sup_{h,\tilde{h} \ s.t. \ h^t=\tilde{h}^t} |u_i(h) - u_i(\tilde{h})| \to 0 \ \text{as} \ t \to \infty
    \]
One-stage Deviation Principle for Infinite Horizon Games (cont.)

• Continuity at infinity condition is satisfied when overall utilities are discounted sum of stage utilities:

\[ u_i = \sum_{t=0}^{\infty} \delta_t^t g_i^t(a^t) \]

where \( g_i^t(a^t) \) are stage utilities, \( 0 \leq \delta_t < 1 \) is discount factor, and stage utilities are uniformly bounded, i.e., there exists \( B \) such that \( \max_{t,a} |g_i^t(a^t)| < B \)

• Theorem:

• Consider infinite-horizon game, \( G^\infty \), that is continuous at infinity. Then, one-stage deviation principle holds, i.e., strategy profile \( s^* \) is SPE if and only if for all \( i, h^t, \) and \( t, \) we have

\[ u_i(s_i^*, s_{-i}^*|h^t) \geq u_i(s_i, s_{-i}^*|h^t) \]

for all \( s_i \) satisfying

\[ s_i(h^t) \neq s_i^*(h^t), \]

\[ s_{|h^t}(h^{t+k}) = s_{|h^t}(h^{t+k}), \text{ for all } k > 0, \text{ and all } h^{t+k} \in G(h^t) \]
Examples: Value of Commitment

• Consider entry deterrence game, but with different timing

• Note: For consistency, first number is still entrant’s utility

• This implies that incumbent can now commit to fighting (how could it do that?)

• It is straightforward to see that unique SPE now involves incumbent committing to fighting and entrant not entering

• This illustrates value of commitment
Examples: Stackleberg Model of Competition

• Consider variant of Cournot model where firm 1 chooses its quantity $q_1$ first, and firm 2 chooses its quantity $q_2$ after observing $q_1$
  • Here, firm 1 is Stackleberg leader

• Suppose again that both firms have marginal cost $c$ and inverse demand function is given by $P(Q) = \alpha - \beta Q$, where $Q = q_1 + q_2$, and $\alpha > c$

• This is dynamic game, so we should look for SPE

• Backward induction (this is not finite game, but all we have seen so far applies to infinite games too)

• Look at subgame indexed by firm 1 quantity choice, $q_1$

• Then firm 2’s maximization problem is essentially the same as before

$$\max_{q_2 > 0} (\alpha - \beta (q_1 + q_2) - c)q_2$$
Stackleberg Competition (cont.)

• This gives best response

\[ q_2 = \frac{\alpha - c - \beta q_1}{2\beta} \]

• Now difference is that firm 1 will choose \( q_1 \) recognizing that firm 2 will respond with above best response function

• Firm 1 is Stackleberg leader and firm 2 is follower

• This means firm 1’s problem is

\[
\max_{q_1 \geq 0} (P(q_1 + q_2) - c)q_1 \\
\text{s.t. } q_2 = \frac{\alpha - c - \beta q_1}{2\beta}
\]
Stackleberg Competition (cont.)

• This gives

\[ q_1^s = \frac{\alpha - c}{2\beta} \]

• And thus

\[ q_2^s = \frac{\alpha - c}{4\beta} < q_1^s \]

• Why lower output for follower?

• Total output is

\[ Q = q_1^s + q_2^s = \frac{3(\alpha - c)}{4\beta} \]

which is greater than Cournot output. Why?
Multiplicity of Subgame Perfect Equilibria

• Question:
  • What happens if in some subgame more than one action is optimal? Consider all optimal actions and trace back implications of each in all of longer subgames

  • Agent 2’s optimal strategies in this game are given by $FHK, FIK, GHK$, and $GIK$
Multiplicity of Subgame Perfect Equilibria (Cont.)

• Now consider agent 1’s optimal strategies for every combination of optimal actions for agent 2:

<table>
<thead>
<tr>
<th>2’s opt. str.</th>
<th>1’s BR</th>
<th>6 SPE’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>FHK</td>
<td>C</td>
<td>(C, FHK)</td>
</tr>
<tr>
<td>FIK</td>
<td>C</td>
<td>(C, FIK)</td>
</tr>
<tr>
<td>GHK</td>
<td>C,D,E</td>
<td>(C, GHK)</td>
</tr>
<tr>
<td>GIK</td>
<td>D</td>
<td>(D, GHK)</td>
</tr>
</tbody>
</table>

1’s BR = BR (Best Response)
Questions?
Acknowledgement

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  - Asu Ozdaglar [MIT 6.254]