ECE700.07: Game Theory with Engineering Applications

Lecture 7: Bargaining Games

Seyed Majid Zahedi
Outline

• Stahl Bargaining Model
• Rubinstein Bargaining Model with Alternating Offers
• Nash Bargaining Solution
• Relation of Axiomatic and Strategic Model

• Readings:
Ultimatum Game

• Two agents use following procedure to split $c$ dollars
  • Agent 1 offers agent 2 some amount $x \leq c$
  • If agent 2 accepts, outcome is $(c - x, x)$
  • If agent 2 rejects, outcome is $(0, 0)$
  • Each agent cares only about amount of money she receives

• What is SPE for this game?
SPE of the Ultimatum Game

• What is optimal action of agent 2 for each such subgame?
  • If $x > 0$, then Yes
  • If $x = 0$, then indifferent between Yes and No

• How many different optimal strategies does agent 2 have?
  • (1) Yes for all $x \geq 0$
  • (2) Yes if $x > 0$ and No if $x = 0$
SPE of Ultimatum Game

• What is agent 1’s optimal strategy for each of these strategies?
  • For (1): agent 1’s optimal offer is \( x = 0 \)
  • For (2): agent 1’s optimal offer is
    • \( x = 0 \rightarrow 0 \)
    • \( x > 0 \rightarrow c - x \quad \max_{x>0}(c - x) \Rightarrow \) no optimal solution
    • No offer of agent 1 is optimal!

• Unique SPE is: Agent 1 offers 0 and agent 2 accepts all offers
SPE of the Ultimatum Game (cont.)

• What if amount of money available is in multiples of cent?
  • There are 2 SPE’s instead of only one:
    • Agent 1 offers 0, and agent 2 says Yes to all offers
    • Agent 1 offers 1 cent, and agent 2 says Yes to all offers except 0

• Show that for every $\bar{x} \in [0, c]$, there exists NE in which 1 offers $\bar{x}$
  • What is agent 2’s optimal strategy?
Bargaining Game

• Assume now that $c = 1$
• Let $x = (x_1, x_2)$ with $x_1 + x_2 = 1$ denote allocation in 1st part
• Let $y = (y_1, y_2)$ with $y_1 + y_2 = 1$ denote allocation in 2nd part
Backward Induction for Bargaining Game

- Second part is ultimatum game in which agent 2 moves first
- This has unique subgame perfect equilibrium given by
  - Agent 2 offers nothing to agent 1 and agent 1 accepts all offers
- In every SPE, agent 2 obtains all pie
- Last Mover’s Advantage
  - In every SPE, agent who makes offer in last period obtains all pie
Finite Horizon Game with Alternating Offers

- Suppose that bargaining takes time and time is valuable
- Agents alternate proposals, future discounted using constant discount factor $0 < \delta_i < 1$ at each period
- For two periods
Backward Induction for Bargaining Game

• In (1), unique SPE is:
  • Agent 2 offers (0,1) and agent 1 accepts all proposals, outcome is (0, $\delta_2$)

• In (2):
  • $N \rightarrow (0, \delta_2)$ and $Y \rightarrow (x_1, x_2)$
  • Two strategies:
    • $Y$ if $x_2 \geq \delta_2$ and $N$ otherwise
    • $Y$ if $x_2 > \delta_2$ and $N$ otherwise

• In (3):
  • Agent 1’s optimal strategy is $(1 - \delta_2, \delta_2)$
Backward Induction for Bargaining Game (cont.)

• Unique SPE of this game is:
  • Agent 1’s initial proposal \((1 - \delta_2, \delta_2)\)
  • Agent 2 accepts all proposals where \(x_2 \geq \delta_2\) and rejects all \(x_2 < \delta_2\)
  • Agent 2 proposes \((0,1)\) after any history in which she rejects 1’s proposal
  • Agent 1 accepts all proposals of Agent 2 (after history in which 2 rejects 1’s opening proposal)

• Outcome of game is:
  • Agent 1 proposes \((1 - \delta_2, \delta_2)\)
  • Agent 2 accepts
  • Resulting utilities: \((1 - \delta_2, \delta_2)\)

• Desirability of earlier agreement yields positive payoff for agent 1
Stahl’s Bargaining Model: Finite Horizon

• 2 periods: $(1 - \delta_2)$
• 3 periods: $(1 - \delta_2) + \delta_1 \delta_2$
• 4 periods: $(1 - \delta_2)(1 + \delta_1 \delta_2)$
• 5 periods: $(1 - \delta_2)(1 + \delta_1 \delta_2) + \delta_1 \delta_2$
• $2k$ periods: $(1 - \delta_2) \left( \frac{1 - (\delta_1 \delta_2)^k}{1 - \delta_1 \delta_2} \right)$
• $2k+1$ periods: $(1 - \delta_2) \left( \frac{1 - (\delta_1 \delta_2)^k}{1 - \delta_1 \delta_2} \right) + (\delta_1 \delta_2)^k$

• Taking limit as $k \to \infty$, we see that agent 1 gets $x_1^* = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$ at SPE
Rubinstein’s Infinite Horizon Bargaining Model

• Suppose agent can alternate offers forever

• There are two types of terminal histories
  • Every offer gets rejected: \((x^1, N, x^2, N, ..., x^t, N, ...)\)
  • At period \(t\), one agent accepts her offer \((x^1, N, x^2, N, ..., x^t, Y)\)

• This game does not have finite horizon, so we cannot use backward induction

• We will instead guess strategy profile and verify SPE using one-stage deviation principle
Rubinstein’s Infinite Horizon Bargaining Model (cont.)

• Consider following strategy profile $s^*$
  
  • Agent 1 proposes $x^*$ and accepts $y$ if and only if $y \geq y_1^*$
  
  • Agent 2 proposes $y^*$ and accepts $x$ if and only if $x \geq x_2^*$

\[
x^* = (x_1^*, x_2^*), \quad x_1^* = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \quad x_2^* = \frac{\delta_2 (1 - \delta_1)}{1 - \delta_1 \delta_2}
\]

\[
y^* = (y_1^*, y_2^*), \quad y_1^* = \frac{\delta_1 (1 - \delta_2)}{1 - \delta_1 \delta_2}, \quad y_2^* = \frac{1 - \delta_1}{1 - \delta_1 \delta_2}
\]
Rubinstein’s Infinite Horizon Bargaining Model (cont.)

• First note that this game has 2 types of subgames:
  • (1) first move is offer
  • (2) first move is response to offer

• For (1), suppose offer made by agent 1
  • Fix agent 2’s strategy
  • If agent 1 adopts $s^*$, agent 2 accepts, agent 1 gets $x_1^*$
  • If agent 1 offers $> x_2^*$, agent 2 accepts leading to lower utility than $x_2^*$ for agent 1
  • If agent 1 offers $< x_2^*$, agent 2 rejects, offers $y_1^*$, agent 1 accepts, leading to utility of $\delta_1 y_1^* < x_1^*$

• For (2), suppose agent 1 is responding
  • Fix agent 2’s strategy and denote by $(y_1, y_2)$ offer to which agent 1 is responding
  • If agent 1 adopts $s^*$, she accepts offer if $y_1 \geq y_1^*$
  • If agent 1 rejects some offer $y_1 \geq y_1^*$, agent 1 will get $\delta_1 x_1^* = y_1^*$, thus she cannot increase her utility

• Hence $s^*$ is SPE (in fact unique SPE, check GT, Section 4.4.2 to verify)
Rubinstein’s Infinite Horizon Bargaining Model (cont.)

• To gain more insight into resulting allocation, assume for simplicity that $\delta_1 = \delta_2$
  
  • If agent 1 moves first, division is $\left( \frac{1}{1-\delta}, \frac{\delta}{1-\delta} \right)$
  • If agent 2 moves first, division is $\left( \frac{\delta}{1-\delta}, \frac{1}{1-\delta} \right)$

• First mover’s advantage is related to impatience of agents (i.e., discount factor $\delta$)
  
  • If $\delta \to 1$, FMA disappears and outcome tends to $\left( \frac{1}{2}, \frac{1}{2} \right)$
  • If $\delta \to 0$, FMA dominates and outcome tends to $(1,0)$
Alternative Bargaining Model: Nash’s Axiomatic Model

- Bargaining problems represent situations in which:
  - There is conflict of interest about agreements
  - Individuals have possibility of concluding mutually beneficial agreement
  - No agreement may be imposed on any individual without her approval
- Strategic or noncooperative model involves explicitly modeling bargaining process (i.e., game form)
- Axiomatic model involves considering only set of outcomes that satisfy “reasonable” properties
- This approach was proposed by Nash in his 1950 paper, where he states: “One states as axioms several properties that would seem natural for the solution to have and then one discovers that axioms actually determine the solution uniquely.”
- First question to answer is: What are some reasonable axioms?
Nash’s Axiomatic Model

• **Example:**
  • Suppose 2 agents with identical preferences must split $1
  • They could discard some portion of $1
  • If no agreement is reached, then agents do not receive anything
  • We then expect
    • Agents to agree (Efficiency)
    • Each to obtain half (Symmetry)

• Let $X$ denote set of possible agreements and $D$ denote disagreement outcome

• For example, we may have
  $$X = \{(x_1, x_2) | x_1 + x_2 \leq 1, x_i \geq 0, i = 1, 2\}, D = (0,0)$$
Nash’s Axiomatic Model (cont.)

- Define $u_i$ to denote agent $i$’s utility function over $X \cup \{D\}$
- Denote set of possible utilities by set $U$ defined as
  - $U = \{(v_1, v_2) \mid u_1(x) = v_1, u_2(x) = v_2 \text{ for some } x \in X\}$
  - $d = (u_1(D), u_2(D))$
- Bargaining Problem (BP) is pair $(U, d)$ where $U \subset \mathbb{R}^2$ and $d \in U$ if
  - $U$ is convex and compact set
  - There exists some $v \in U$ such that $v > d$ (i.e., $v_i > d_i, \forall i$)
- Denote set of all possible BPs by $\mathcal{B}$
- Bargaining solution is function $f: \mathcal{B} \rightarrow U$
Example: Dividing $\$1$

- Let $X = \{(x_1, x_2) \mid x_1 + x_2 \leq 1, \ x_i \geq 0, \ i = 1, 2\}, \ D = (0,0)$
- Suppose $u_i(x_i)$ is concave (users are risk-averse) and $u_i(0) = 0$
- Define $U = \{(v_1, v_2) \mid (v_1, v_2) = (u_1(x_1), u_2(x_2)), (x_1, x_2) \in X\}$
  - It is compact and convex
  - It contains $d = (u_1(0), u_2(0)) = (0,0)$
  - There is point $v \in U$ such that $v_i > d_i$ for $i = 1, 2$
  - Therefore, $(U, d)$ is BP
Some Definitions

• BP $(U, d)$ is symmetric if
  • $d_1 = d_2$
  • $(v_1, v_2) \in U$ if and only if $(v_2, v_1) \in U$

• $(U', d')$ is obtained from BP $(U, d)$ by transformations $(\alpha, \beta)$, where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, if
  • $d'_i = \alpha_i d_i + \beta_i$ for $i = 1, 2$
  • $U' = \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) \in \mathbb{R}^2: (v_1, v_2) \in U\}$
  • It could be checked that if $\alpha_i > 0$ for $i = 1, 2$, then $(U', d')$ is BP too
Axioms

• **Symmetry (SYM)**
  - If agents are indistinguishable, agreement should not discriminate between them
  - If BP $(U, d)$ is symmetric, then $f_1(U, d) = f_2(U, d)$

• **Pareto Efficiency (PAR)**
  - Agents never agree on outcome $u$ when there is outcome $t$ in which they are both better off
  - If $(U, d)$ is BP, $v \in U$, $t \in U$, and $t_i > v_i$ for $i = 1, 2$, then $f(U, d) \neq v$
Axioms (cont.)

• **Invariance to Equivalent Payoff Representations (INV)**
  • Transformation of utility function that maintains the same ordering over preferences (such as linear transformation) should not alter outcome of bargaining process
  • If BP \((U', d')\) is obtained from BP \((U, d)\) by transformation \((\alpha, \beta)\), with \(\alpha > 0\), then \(f_i(U', d') = \alpha_i f_i(U, d) + \beta_i\) for \(i = 1, 2\)

• **Independence of Irrelevant Alternatives (IIA)**
  • If when all alternatives in \(U\) are available, the agents agree on outcome in smaller set \(U'\), then we require that agents agree on the same outcome when only alternative in \(U'\) are available
  • If \((U, d)\) and \((U', d)\) be two BPs such that \(U' \subset U\), and if \(f(U, d) \in U'\), then \(f(U', d) = f(U, d)\)
Nash Bargaining Solution

- \( f^N(U, d) \) is Nash Bargaining Solution (NBS) if and only if
  \[
  f^N(U, d) = \arg \max_{(d_1,d_2)\subseteq (v_1,v_2) \in U} (v_1 - d_1)(v_2 - d_2)
  \]

- Since \( U \) is compact and objective is continuous, \textbf{NBS exists}

- Objective is strictly quasi-concave which means \textbf{NBS is unique}
Nash Bargaining Solution (cont.)

• **Proposition:**
  • NBS is unique bargaining solution that satisfies all 4 axioms

• **Proof has 2 steps:**
  • NBS satisfies all 4 axioms
  • If bargaining solution satisfies all 4 axioms, it is NBS

• **Step 1:**
  • **Pareto efficiency:** It follows from objective being increasing in $v_1$ and $v_2$
  • **Symmetry:** If $d_1 = d_2$ and $f^N(U, d) = (v_1^*, v_2^*)$ be NBS, then it can be seen that $(v_2^*, v_1^*)$ is also optimal solution of maximization, which means by uniqueness of NBS, we must have $v_1^* = v_2^*$
Nash Bargaining Solution (cont.)

- Independence of irrelevant alternatives: If $U' \subset U$, then objective function value at $f^N(U, d)$ is greater than or equal to that at $f^N(U', d)$. If $f^N(U, d) \in U'$, then objective function values must be equal, which means $f^N(U, d)$ is optimal for $U'$ and by uniqueness of UBS, $f^N(U, d) = f^N(U', d)$

- Invariance to equivalent payoff representations: It follows by performing change of variable $v_i' = \alpha_i v_i + \beta_i$, for $i = 1, 2$
Nash Bargaining Solution (cont.)

• **Step 2:**
  
  • Let $f(U,d)$ be bargaining solution satisfying all 4 axioms
  • Let $f^N(U,d) = v^*$
  • Define $(\alpha, \beta)$ such that $\alpha_i v_i^* + \beta_i = 1/2$ and $\alpha_i d_i + \beta_i = 0$ for $i = 1, 2$
  • Define $(U', 0)$ to be transformation of $(U, d)$ by $(\alpha, \beta)$
  • Since $\alpha > 0$, $U'$ is BP
  • Since $f$ and $f^N$ satisfy INV, we have
    
    • $f_i(U', 0) = \alpha_i f(U, d) + \beta_i$ for $i = 1, 2$
    • $f_i^N(U', 0) = \alpha_i f_i^N(U, d) + \beta_i = 1/2$ for $i = 1, 2$
    • If we prove $f_i(U', 0) = 1/2$ for $i = 1, 2$, then $f(U, d) = f^N(U, d)$
Nash Bargaining Solution (cont.)

• Let us show that there is no $v \in U'$ such that $v_1 + v_2 > 1$
  • Assume that there is $v \in U'$ such that $v_1 + v_2 > 1$
  • Let $(t_1, t_2) = (1 - \lambda)(1/2, 1/2) + \lambda(v_1, v_2)$ for $\lambda \in (0,1)$
  • Since $U'$ is convex, we have $t \in U'$
  • We can choose $\lambda$ sufficiently small so that $t_1t_2 > \frac{1}{4} = f_N(U', 0)$, but this contradicts optimality of $f_N(U', 0)$
Nash Bargaining Solution (cont.)

- Since $U'$ is bounded, we can find rectangle $U''$ symmetric with respect to the line $v_1 = v_2$, such that $U' \subset U''$ and $(1/2, 1/2)$ is on boundary of $U''$

- By PAR and SYM, $f(U'', 0) = (1/2, 1/2)$
- By IIA, since $U' \subset U''$, we have $f(U', 0) = (1/2, 1/2)$
Example: Dividing $1$

- Let $X = \{(x_1, x_2) \mid x_1 + x_2 \leq 1, \ x_i \geq 0, \forall i\}, D = (0,0)$
- Suppose $u_i(x_i)$ is concave (users are risk-averse) and $u_i(0) = 0$
- If $u_1 = u_2$, by PAR and SYM, unique NBS is $(1/2,1/2)$
- If agent 2 is more risk averse, i.e., $v_1 = u$ and $v_2 = h \circ u$, where $h: \mathbb{R} \to \mathbb{R}$ is increasing concave function with $h(0) = 0$
- NBS is solution of $\max_{0 \leq z \leq 1} u(z)h(u(1-z))$
- By first order optimality
  - $u'(z)h(u(1-z)) = u(z)h'(u(1-z))u'(1-z)$
- In class assignment (Quiz): show that when agent 2 is more risk averse, agent 1’s share increases
  - Hint: Since $h$ is concave increasing function, and $h(0) = 0$, then $h'(t) \leq \frac{h(t)}{t}$ for $t > 0$
Questions?
Acknowledgement

• This lecture is a slightly modified version of one prepared by
  • Asu Ozdaglar [MIT 6.254]